Scientific Day in Memory of Prof. Mila Nikolova

15 October 2018

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International Conference on Image Processing — ICIP'99

Volume 2 — TUESDAY, OCTOBER 26 sessions

26AS1: STOCHASTIC GEOMETRIC APPROACH TO IMAGE ANALYSIS

Organizer: Prof. Alfred O. Hero III, Univ. of Michigan
Prof. Hamid Krim, North Carolina State University


THE PEAK

Seminar

Image Restoration by Minimizing Cost-Functions with Non-smooth Data-Fidelity Terms and Application to the Processing of Outliers

Professor Mila Nikolova
ENST, France

on January 18, 2002 (Friday),

4:00 - 5:00pm

Room 517, Meng Wah Complex

All are welcome

THE UNIVERSITY OF HONG KONG

Department of Mathematics

JOINT FRANCE/HONG KONG IMAGE PROCESSING SEMINAR

2 November 2002 (Saturday)

9:30 am - 12:30 pm

517 Meng Wah Complex, HKU

9:30 – 10:20	Dr. Pierre Kornprobst INRIA, Odyssee Lab, France *About some variational problem in image processing*
10:20 – 11:00	Dr. Siu Pang Yung Department of Mathematics, The University of Hong Kong *Discrete periodic wavelet filters of trigonometric vanishing moments*
11:00 – 11:10	*Coffee Break*
11:10 – 11:50	Dr. Mila Nikolova ENST, France *Comparison of the main forms of half-quadratic regularization*
11:50 – 12:30	Dr. Michael K. Ng Department of Mathematics, The University of Hong Kong *High-resolution image reconstruction and color imaging*

Supported by the France/Hong Kong Joint Research Scheme

Joint France-Hong Kong Image Processing Workshop

2 February 2007

Time: **2:00 p.m. - 6:00 p.m.**
Venue: **FSC1217, Fong Shu Chuen Building**
Ho Sin Hang Campus
Hong Kong Baptist University
Program

Speakers:

Francois Malgouyres (Université Paris 13)

Michael Ng (Hong Kong Baptist University)

Mila Nikolova (CMLA ENS de Cachan)

Chong Sze Tong (Hong Kong Baptist University)

Poon Chi Yuen (Hong Kong Baptist University)

Organized by
Centre for Mathematical Imaging and Vision (CMIV), Hong Kong Baptist University

Sponsored by
Centre for Mathematical Imaging and Vision (CMIV), Hong Kong Baptist University
The Consulate General France in Hong Kong

- 8 May 2007
  Prof. Kin Hong Wong
  Real-Time Pose Tracking and Model Reconstruction

- January-February 2007
  Prof. Francois Malgouyres
  Lecture Series:
  (i) Defining a Data Fidelity Term by a Polytope: Application to Image Restoration and Compression
  (ii) Projecting onto a Polytope Simplifies Data Distributions: Theory and Some Applications
  (iii) Primal-dual Implementation of the Basis Pursuit Algorithm

- February-April 2007
  Prof. Mila Nikolova
  Lecture Series: Optimization for Image Restoration

- 7 November 2006
  Prof. Ke Chen
  Optimization-based Multilevel Methods for Image Restoration

I Generalities

II Differentiable problems

- without constraints
- with general constraints
- constraints equality
  inequality

III Nonsmooth optimization





GOOD
THINGS
TAKE
TIME


In memory of Mila Nikolova

Robust Tensor Completion and its Applications

Michael K. Ng

Department of Mathematics
Hong Kong Baptist University
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Tensor Decomposition

CANDECOMP/PARAFAC Decomposition:

$$\mathcal{X} = \sum_{i=1}^{r} \lambda_i \mathbf{a}^{i,1} \otimes \cdots \otimes \mathbf{a}^{i,m}$$

The minimal value of r is called the rank of $\mathcal{A}$.

Tensor Decomposition

Tucker Decomposition:

$$\mathcal{X} = \mathcal{G} \times \mathbf{A}_1 \times \mathbf{A}_2 \cdots \times \mathbf{A}_m$$

$$\mathcal{X} = \sum_{i_1=1}^{r_1} \cdots \sum_{i_m=1}^{r_m} g_{i_1,i_2,\cdots,i_m} \mathbf{a}^{i_1,1} \otimes \cdots \otimes \mathbf{a}^{i_m,m}$$

It can be obtained by using singular value decomposition to each unfolded matrix $\mathbf{X}_{i_j}$ from $\mathcal{X}$. The Tucker rank is $(rank(\mathbf{X}_1), rank(\mathbf{X}_2), \cdots, rank(\mathbf{X}_m)) = (r_1, r_2, \cdots, r_m)$.

Low-dimensional Structure

Data in many real applications exhibit low-dimensional structures due to local regularities, global symmetries, repetitive patterns, redundant sampling, ... (low-dimensional structure $\rightarrow$ low-rank data matrices)

Example

Customer/Item	I	II	III	IV	$\cdots$
A	5	1	?	?	$\cdots$
B	?	2	3	?	$\cdots$
C	?	?	4	2	$\cdots$
D	1	?	?	?	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

For example (Netflix Challenge 2009), it is about 0.5 million users and about 18,000 movies

Matrix Completion

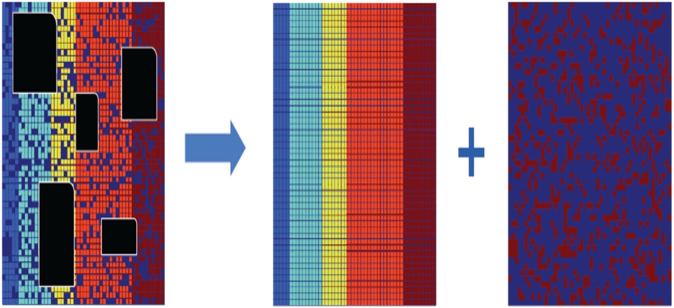
$$\min_{\mathbf{X}} \ rank(\mathbf{X}) \quad \text{subject to} \quad P_\Omega(\mathbf{X}) = P_\Omega(\mathbf{M})$$

Example

Matrix RPCA

$$\min_{\mathbf{X}} \; rank(\mathbf{X}) + \lambda \|\mathbf{E}\|_0 \quad \text{subject to} \quad \mathbf{X} + \mathbf{E} = \mathbf{M}$$

Example



Robust Matrix Completion

$$\min_{\mathbf{X}} \; rank(\mathbf{X}) + \lambda \|\mathbf{E}\|_0 \quad \text{subject to} \quad P_\Omega(\mathbf{X}+\mathbf{E}) = P_\Omega(\mathbf{M})$$

Low Rank Matrix Recovery

- Matrix Completion

$$\min_{\mathbf{X}} \ rank(\mathbf{X}) \quad \text{subject to} \quad P_\Omega(\mathbf{X}) = P_\Omega(\mathbf{M})$$

- Matrix RPCA

$$\min_{\mathbf{X}} \ rank(\mathbf{X}) + \lambda \|\mathbf{E}\|_0 \quad \text{subject to} \quad \mathbf{M} = \mathbf{X} + \mathbf{E}$$

- Robust Matrix Completion

$$\min_{\mathbf{X}} \ rank(\mathbf{X}) + \lambda \|\mathbf{E}\|_0 \quad \text{subject to} \quad P_\Omega(\mathbf{M}) = P_\Omega(\mathbf{X} + \mathbf{E})$$

Low Rank Matrix Recovery

- Matrix Completion

$$\min_{\mathbf{X}} \ \|\mathbf{X}\|_* \quad \text{subject to} \quad P_\Omega(\mathbf{X}) = P_\Omega(\mathbf{M})$$

- Matrix RPCA

$$\min_{\mathbf{X}} \ \|\mathbf{X}\|_* + \lambda\|\mathbf{E}\|_1 \quad \text{subject to} \quad \mathbf{M} = \mathbf{X} + \mathbf{E}$$

- Robust Matrix Completion

$$\min_{\mathbf{X}} \ \|\mathbf{X}\|_* + \lambda\|\mathbf{E}\|_1 \quad \text{subject to} \quad P_\Omega(\mathbf{M}) = P_\Omega(\mathbf{X} + \mathbf{E})$$

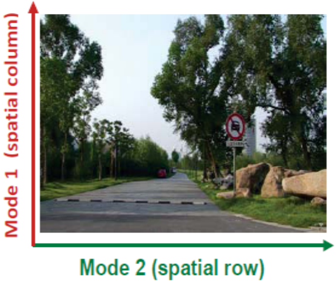
Nuclear norm $\|\cdot\|_*$: sum of singular values (convex envelop of rank)

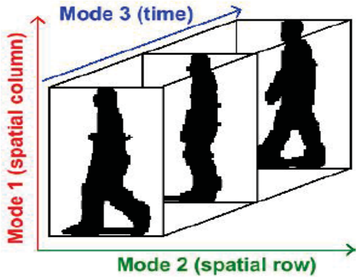
Low Rank Matrix Recovery Results

- (RPCA) Candes, E. J., Li, X., Ma, Y., and Wright, J. Journal of the ACM, 58(3):173, 2011.
- (Matrix Completion) Recht, B. Journal of Machine Learning Research, 12(4):34133430, 2011.
- (Matrix Completion) Chen, Y. IEEE Transactions on Information Theory, 61(5):29092923, 2013.

Low Rank Tensor Recovery

Data are usually in multi-dimensional array.





"Vectorization" probably break the inherent structures and correlations in the original data.

Low Rank Tensor Recovery

- Tensor Completion

$$\min_{\mathcal{X}} \ rank(\mathcal{X}) \quad \text{subject to} \quad P_\Omega(\mathcal{X}) = P_\Omega(\mathcal{M})$$

- Tensor Robust PCA

$$\min_{\mathcal{X}} \ rank(\mathcal{X}) + \lambda \|\mathcal{E}\|_0 \quad \text{subject to} \quad \mathcal{M} = \mathcal{X} + \mathcal{E}$$

- Robust Tensor Completion

$$\min_{\mathcal{X}} \ rank(\mathcal{X}) + \lambda \|\mathcal{E}\|_0 \quad \text{subject to} \quad P_\Omega(\mathcal{M}) = P_\Omega(\mathcal{X} + \mathcal{E})$$

Low Rank Tensor Recovery

- CP decomposition/rank cannot be computed efficiently
- Matrix rank can be replaced by matrix nuclear norm (the sum of singular values), it is a convex envelope
- Replace Tucker rank by the sum of nuclear norms of unfolding tensors, interdependent matrix trace norm is involved
- The use of the sum of nuclear norms of unfolding matrices of a tensor may be challenged since it is suboptimal[1]
- The tensor trace norm (the average of trace norms of unfolding matrices) is not a tight convex relaxation of the tensor rank (the average rank of unfolding matrices) [2]

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[1] C. Mu, B. Huang, J. Wright, and D. Goldfarb. Square deal: Lower bounds and improved relaxations for tensor recovery. In ICML, pages 7381, 2014.

[2] B. Romera-Paredes and M. Pontil. A new convex relaxation for tensor completion. In Adv. Neural Inf. Process. Syst., pages 29672975, 2013.

t-SVD Decomposition

A third-order tensor of size $n_1 \times n_2 \times n_3$ can be viewed as an $n_1 \times n_2$ matrix of tubes which lie in the third-dimension. [Kilmer, M. E. and Martin, C. D. Linear Algebra & Its Applications, 435(3):641658, 2011]

t-SVD Decomposition

Definition: The t-product $\mathcal{A} * \mathcal{B}$ of $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ and $\mathcal{B} \in \mathbb{R}^{n_2 \times n_4 \times n_3}$ is a tensor $\mathcal{C} \in \mathbb{R}^{n_1 \times n_4 \times n_3}$ whose (i, j)th tube is given by

$$\mathcal{C}(i, j, :) = \sum_{k=1}^{n_2} \mathcal{A}(i, k, :) * \mathcal{B}(k, j, :),$$

where $*$ denotes the circular convolution between two tubes of same size.

The tube at (i, k) position in $\mathcal{A}$ convolutes with the tube at (k, j) position in $\mathcal{B}$. Both have sizes n_3. Put all the correlations at (i, j) position in $\mathcal{C}$.

The multiplication of between the scalars is replaced by circular convolution between the tubes.

t-SVD Decomposition

Definition: The identity tensor $\mathcal{I} \in \mathbb{R}^{n \times n \times n_3}$ is defined to be a tensor whose first frontal slice $\mathcal{I}^{(1)}$ is the $n \times n$ identity matrix and whose other frontal slices $\mathcal{I}^{(i)}, i = 2, \ldots, n_3$ are zero matrices.

Definition: The conjugate transpose of a tensor $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ is the tensor $\mathcal{A}^H \in \mathbb{R}^{n_2 \times n_1 \times n_3}$ obtained by conjugate transposing each of the frontal slice and then reversing the order of transposed frontal slices 2 through n_3, i.e.,

$$\begin{aligned} \left(\mathcal{A}^H\right)^{(1)} &= \left(\mathcal{A}^{(1)}\right)^H, \\ \left(\mathcal{A}^H\right)^{(i)} &= \left(\mathcal{A}^{(n_3+2-i)}\right)^H, \quad i = 2, \ldots, n_3. \end{aligned}$$

t-SVD Decomposition

Definition: A tensor $\mathcal{Q} \in \mathbb{R}^{n \times n \times n_3}$ is orthogonal if it satisfies

$$\mathcal{Q}^H * \mathcal{Q} = \mathcal{Q} * \mathcal{Q}^H = \mathcal{I},$$

where $\mathcal{I}$ is the identity tensor of size $n \times n \times n_3$.

Definition: A tensor $\mathcal{A}$ is called f-diagonal if each frontal slice $\mathcal{A}^{(i)}$ is a diagonal matrix.

t-SVD Decomposition

For $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, the t-SVD of $\mathcal{A}$ is given by

$$\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^H,$$

where $\mathcal{U} \in \mathbb{R}^{n_1 \times n_1 \times n_3}$ and $\mathcal{V} \in \mathbb{R}^{n_2 \times n_2 \times n_3}$ are orthogonal tensors, and $\mathcal{S} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ is a f-diagonal tensor, respectively. The entries in $\mathcal{S}$ are called the singular tubes of $\mathcal{A}$.

t-SVD Decomposition

The tensor tubal-rank, denoted as $rank_t(\mathcal{A})$, is defined as the number of nonzero singular tubes of $\mathcal{S}$, where $\mathcal{S}$ comes from the t-SVD of $\mathcal{A}$, i.e.,

$$rank_t(\mathcal{A}) = \#\{i : \mathcal{S}(i,i,:) \neq \vec{0}\}.$$

It can be shown that it is equal to $\max_i rank(\hat{\mathcal{A}}^{(i)})$ where $\hat{\mathcal{A}}^{(i)}$ is the i-th slice of $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}$ represents a third-order tensor obtained by taking the Discrete Fourier Transform (DFT) of all the tubes along the third dimension of $\mathcal{A}$.

Definition: The tubal nuclear norm of a tensor $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, denoted as $\|\mathcal{A}\|_{\mathrm{TNN}}$, is the nuclear norm of all the frontal slices of $\hat{\mathcal{A}}$.

Low Tubal Rank Tensor Recovery

- Tensor Completion

$$\min_{\mathcal{X}} \; rank(\mathcal{X}) \quad \text{subject to} \quad P_\Omega(\mathcal{X}) = P_\Omega(\mathcal{M})$$

- Tensor Robust PCA

$$\min_{\mathcal{X}} \; rank(\mathcal{X}) + \lambda \|\mathcal{E}\|_0 \quad \text{subject to} \quad \mathcal{M} = \mathcal{X} + \mathcal{E}$$

- Robust Tensor Completion

$$\min_{\mathcal{X}} \; rank(\mathcal{X}) + \lambda \|\mathcal{E}\|_0 \quad \text{subject to} \quad P_\Omega(\mathcal{M}) = P_\Omega(\mathcal{X} + \mathcal{E})$$

Low Tubal Rank Tensor Recovery (Relaxation)

- Tensor Completion

$$\min_{\mathcal{X}} \|\mathcal{X}\|_{\mathsf{TNN}} \quad \text{subject to} \quad P_\Omega(\mathcal{X}) = P_\Omega(\mathcal{M})$$

- Tensor Robust PCA

$$\min_{\mathcal{X}} \|\mathcal{X}\|_{\mathsf{TNN}} + \lambda\|\mathcal{E}\|_1 \quad \text{subject to} \quad \mathcal{M} = \mathcal{X} + \mathcal{E}$$

- Robust Tensor Completion

$$\min_{\mathcal{X}} \|\mathcal{X}\|_{\mathsf{TNN}} + \lambda\|\mathcal{E}\|_1 \quad \text{subject to} \quad P_\Omega(\mathcal{M}) = P_\Omega(\mathcal{X} + \mathcal{E})$$

Can we recover low-tubal-rank tensor from partial and grossly corrupted observations exactly ?

Tensor Incoherence Conditions

Assume that $rank_t(\mathcal{L}_0) = r$ and its t-SVD $\mathcal{L}_0 = \mathcal{U} * \mathcal{S} * \mathcal{V}^H$. $\mathcal{L}_0$ is said to satisfy the tensor incoherence conditions with parameter $\mu > 0$ if

$$\max_{i=1,\cdots,n_1} \|\mathcal{U}^H * \vec{e}_i\|_F \leq \sqrt{\frac{\mu r}{n_1}},$$

$$\max_{j=1,\cdots,n_2} \|\mathcal{V}^H * \mathring{\boldsymbol{e}}_j\|_F \leq \sqrt{\frac{\mu r}{n_2}},$$

and (joint incoherence condition)

$$\|\mathcal{U} * \mathcal{V}^H\|_\infty \leq \sqrt{\frac{\mu r}{n_1 n_2 n_3}}.$$

Tensor Incoherence Conditions

The **column basis**, denoted as $\vec{e}_i$, is a tensor of size $n_1 \times 1 \times n_3$ with its $(i, 1, 1)$th entry equaling to 1 and the rest equaling to 0. The **tube basis**, denoted as $\mathring{\mathbf{e}}_k$, is a tensor of size $1 \times 1 \times n_3$ with its $(1, 1, k)$th entry equaling to 1 and the rest equaling to 0.

Low Rank Tensor Recovery

Theorem

Suppose $\mathcal{L}_0 \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ obeys tensor incoherence conditions, and the observation set Ω is uniformly distributed among all sets of cardinality $m = \rho n_1 n_2 n_3$. Also suppose that each observed entry is independently corrupted with probability γ. Then, there exist universal constants $c_1, c_2 > 0$ such that with probability at least $1 - c_1 (n_{(1)} n_3)^{-c_2}$, the recovery of $\mathcal{L}_0$ with $\lambda = 1/\sqrt{\rho n_{(1)} n_3}$ is exact, provided that

$$r \leq \frac{c_r n_{(2)}}{\mu (\log(n_{(1)} n_3))^2} \quad \text{and} \quad \gamma \leq c_\gamma$$

where c_r and c_γ are two positive constants.

$n_{(1)} = \max\{n_1, n_2\}$ and $n_{(2)} = \min\{n_1, n_2\}$

Low Rank Tensor Recovery

Theorem

(Tensor Completion): Suppose $\mathcal{L}_0 \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ *obeys tensor incoherence conditions, and* m *entries of* $\mathcal{L}_0$ *are observed with locations sampled uniformly at random, then there exist universal constants* $c_0, c_1, c_2 > 0$ *such that if*

$$m \geq c_0 \mu r n_{(1)} n_3 (\log(n_{(1)} n_3))^2,$$

$\mathcal{L}_0$ *is the unique minimizer to the convex optimization problem with probability at east* $1 - c_1 (n_{(1)} n_3)^{-c_2}$.

Low Rank Tensor Recovery

The detailed theoretical and numerical results can be found in
https://arxiv.org/abs/1708.00601

Transform-based t-SVD

The first work is given by E. Kernfeld, M. Kilmer and S. Aeron, Tensor tensor products with invertible linear transforms, LAA, Vol 485, pp. 545-570 (2015).

We generalize tensor singular value decomposition by using other unitary transform matrices instead of discrete Fourier/cosine transform matrix.

The motivation is that a lower transformed tubal tensor rank may be obtained by using other unitary transform matrices than that by using discrete Fourier/cosine transform matrix, and therefore this would be more effective for robust tensor completion.

Transform-based t-SVD

The detailed theoretical and numerical results can be found in
http://www.math.hkbu.edu.hk/~mng/RTC.pdf

Summary

- More and more applications involving tensor data
- Theory and Algorithms to be studied