In memory of MILA NIKOLOVA

A Day with Mila in Hongkong: April 28th 2008

Restoration of Manifold-Valued Images

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joint work with

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Outline

1. Motivation
2. Variational Models with First and Second Order Priors
3. Numerical Algorithms
   (a) (Sub)Gradient Descent Algorithm
   (b) Cyclic Proximal Point Algorithm (Inexact)
   (c) Parallel Douglas Rachford Algorithm
   (d) Half Quadratic Minimization
4. Nonlocal Patch Based Methods Using MMSE (with M. Nikolova)
5. Numerical Examples

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40

Image grid: $\mathcal{G} := \{1,\ldots,n_1\} \times \{1,\ldots,n_2\}$

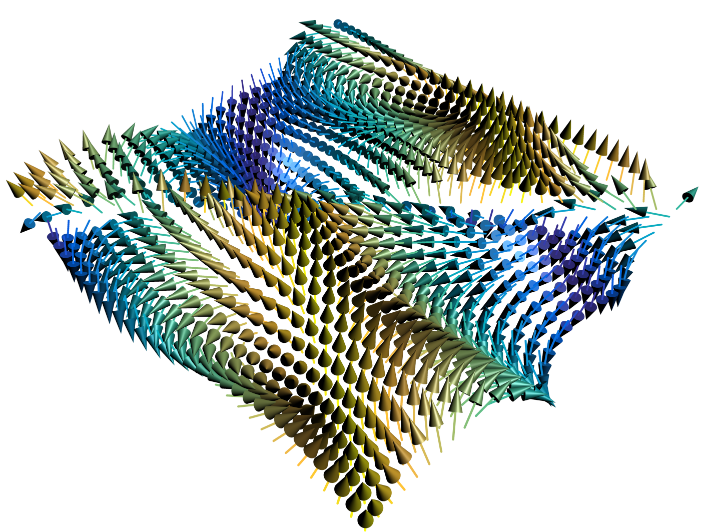
Topic: Restoration of manifold-valued images

$$f : \mathbb{R}^n \supseteq \mathcal{G} \to \mathcal{M}, \quad n \in \{2,3\}$$

where $\mathcal{M}$ is a complete connected finite dimensional Riemannian manifold with distance d.

These are **not** function on manifolds

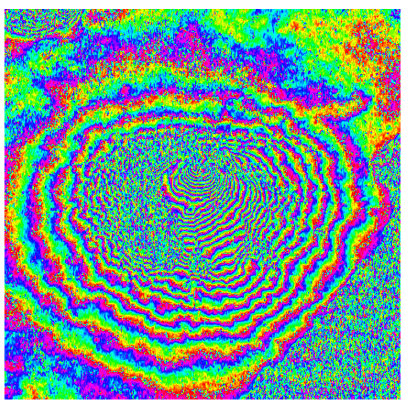
$$f : \mathcal{M} \to \mathbb{R}$$



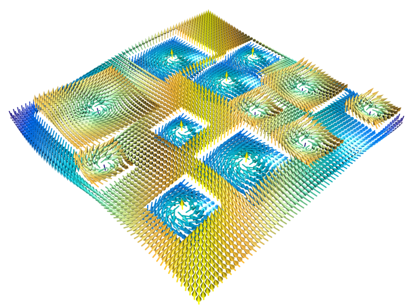
$\mathbb{S}^2$-valued 32×32 image

1. Motivation

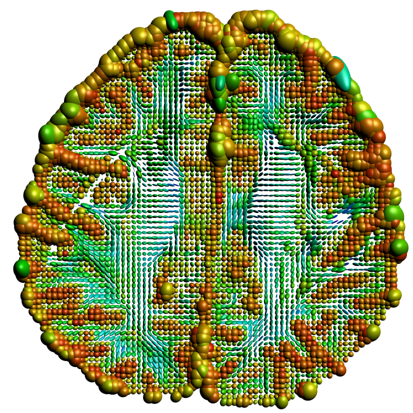
- $\mathbb{S}^1$: Phase space, InSAR imaging, HSI
- $\mathbb{S}^2$: Directions, Chromaticity-Brightness
- $\mathrm{SO}(3)$, $\mathrm{SE}(3)$: Electron Backscattered Diffraction, tracking, motion analysis
- $\mathrm{SPD}(n)$: DT-MRI, covariance matrix information (texture,BCI)



InSAR: $\mathbb{S}^1$



$\mathbb{S}^2$



DT-MRI: SPD(3)

EBSD: $\mathrm{SO}(3)/S$

Image credits: Vesuvius: Rocca, Prati, Guarnierri 1997, Camino project http://cmic.cs.ucl.ac.uk/camino

Reference: R. Bergmann, F. Laus, J. Persch and G. Steidl. Manifold-valued image processing. SIAM News. 50, (8), 1,3 (2017)

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Variational Model for Real-Valued Data: $f, u : \mathcal{G} \to \mathbb{R}$

$$\mathcal{E}(u) := \underbrace{\mathcal{D}(u;f)}_{\text{data term}} + \alpha \underbrace{\mathcal{R}(u)}_{\text{priors}}, \qquad \alpha > 0$$

Data Term: $\mathcal{D}(u;f) := \frac{1}{2}\|f-u\|_2^2 = \sum_{i\in\mathcal{G}} |f_i - u_i|^2$

TV Prior:

$$\mathrm{TV}(u) := \|\nabla u\|_{2,1} = \sum_{i\in\mathcal{G}} \Big(\sum_{j\in\mathcal{N}(i)} |u_j - u_i|^2 \Big)^{\frac{1}{2}},$$

where $\mathcal{N}(i) := \{i+(0,1), i+(1,0)\} \cap \mathcal{G}$

TV_2 Prior:

$$\mathrm{TV}_2(u) := \|\nabla^2 u\|_{2,1}$$

$$= \sum_{i\in\mathcal{G}} \Big(|(D_{xx}u)_i|^2 + |(D_{yy}u)_i|^2 + \frac{1}{4}\left((D_{xy}u)_i + (D_{yx}u)_i\right)^2 \Big)^{\frac{1}{2}}$$

with forward and backward differences defined as

$$(\nabla_x u)_i := \begin{cases} u_{i+(1,0)} - u_i & \text{if } i + (1,0) \in \mathcal{G}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\widetilde{\nabla}_x u)_i := \begin{cases} u_i - u_{i-(1,0)} & \text{if } i \pm (1,0) \in \mathcal{G}, \\ 0 & \text{otherwise.} \end{cases}$$

First order differences (gradients):

$$\nabla := \begin{pmatrix} \nabla_x \\ \nabla_y \end{pmatrix}, \quad \widetilde{\nabla} := \begin{pmatrix} \widetilde{\nabla}_x & \\ \frac{1}{2}\widetilde{\nabla}_y & \frac{1}{2}\widetilde{\nabla}_x \\ & \widetilde{\nabla}_y \end{pmatrix}$$

Second order differences (symmetrized Hessian):

$$\nabla^2 := \widetilde{\nabla}\nabla = \begin{pmatrix} D_{xx} \\ \frac{1}{2}(D_{xy} + D_{yx}) \\ D_{yy} \end{pmatrix}$$

Coupled Priors: $\beta \in (0,1)$

1. **Additive Coupling**:

$$\mathrm{TV}_{1\wedge 2}(u) := \beta \mathrm{TV}(u) + (1-\beta)\mathrm{TV}_2(u)$$

2. **Infimal Convolution**:

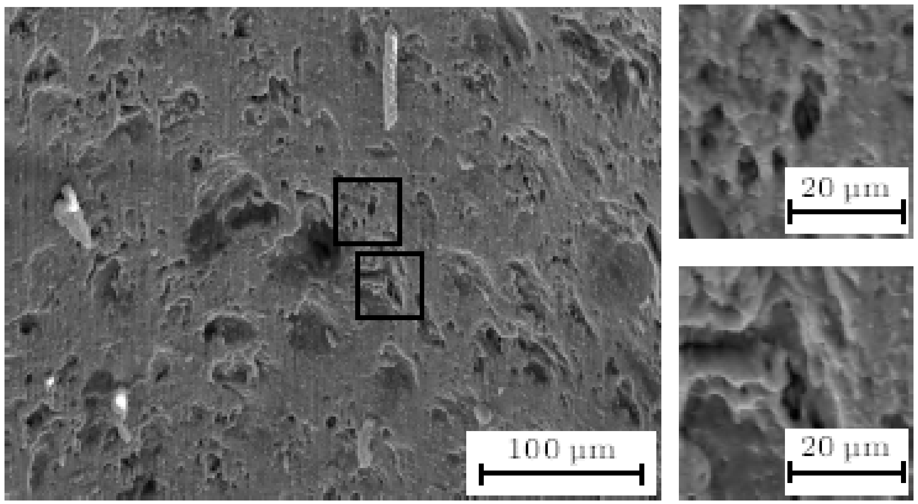
$$\begin{aligned} \mathrm{IC}(u) &:= (\beta \mathrm{TV} \square (1-\beta)\mathrm{TV}_2)(u) \\ &= \inf_{u=v+w} \{\beta \mathrm{TV}(v) + (1-\beta)\mathrm{TV}_2(w)\} \end{aligned}$$

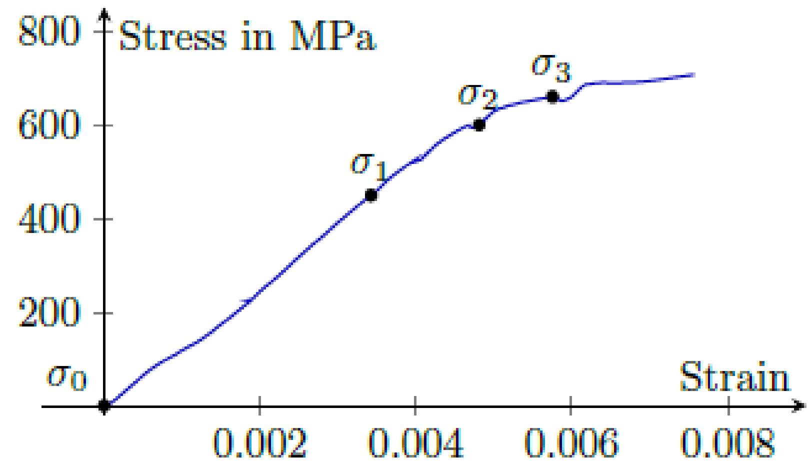
3. **Total Generalized Variation**:

$$\mathrm{TGV}(u) := \inf_a \left\{ \beta \int_\Omega |\nabla u - a| \, \mathrm{d}x + (1-\beta) \int_\Omega |\tilde{\nabla} a| \, \mathrm{d}x \right\}$$

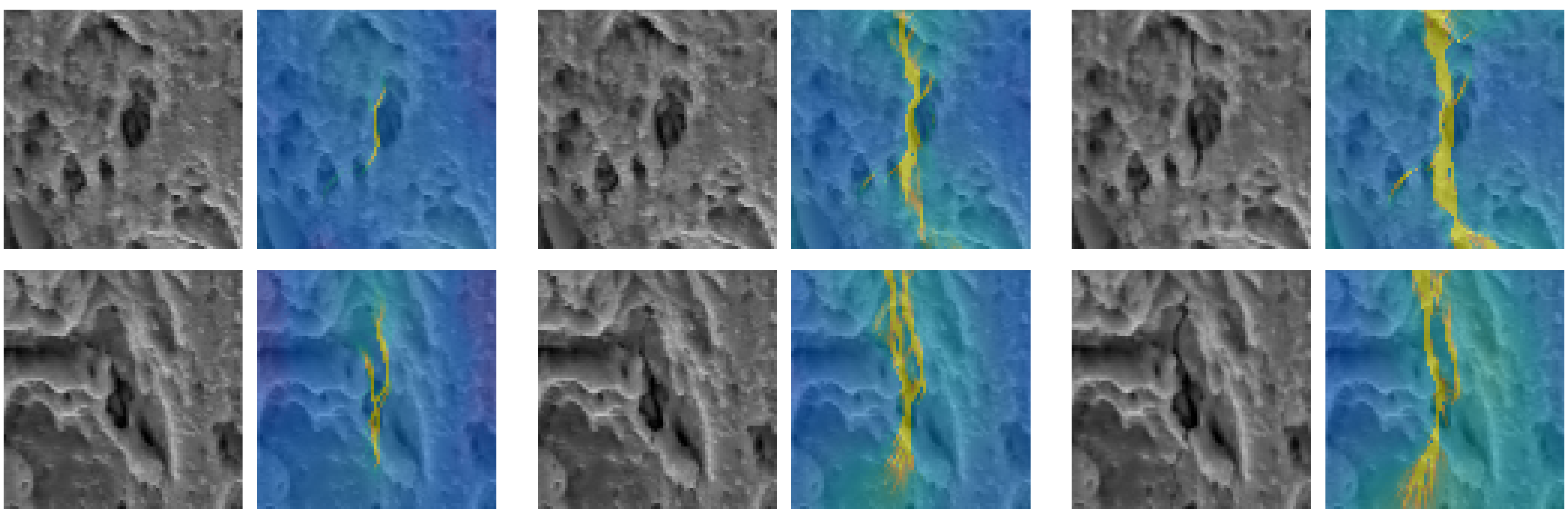
References: Chambolle/Lions 1997, Bredies/Kunisch/Pock 2010, Setzer/Steidl 2008

One of the Best Applications of an OF Model with TGV Prior





450 MPa | 600 MPa | 660 MPa



Early crack detection in materials

New: Images: $f, u : \mathcal{G} \to \mathcal{M}$

Idea: Embedding of $\mathcal{M}$ into $\mathbb{R}^N$ (Theorems of Whitney nad Nash)

1. **Additive Coupling**:

$$J_{\mathrm{Add}}^{\mathrm{ext}}(u) := \frac{1}{2}\|f - u\|_2^2 + \alpha \, \mathrm{TV}_{1\wedge 2}(u) + \iota_{\mathcal{M}^N}(u)$$

2. **Infimal Convolution**:

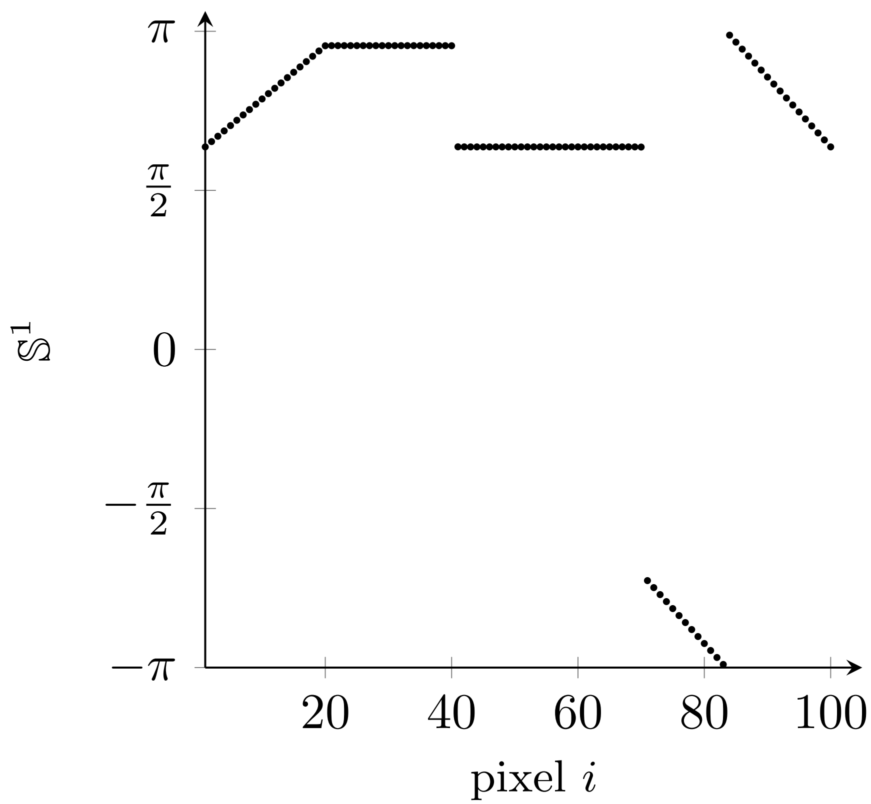
$$J_{\mathrm{IC}}^{\mathrm{ext}}(u) := \frac{1}{2}\|f - u\|_2^2 + \alpha \, \mathrm{IC}(u) + \iota_{\mathcal{M}^N}(u)$$

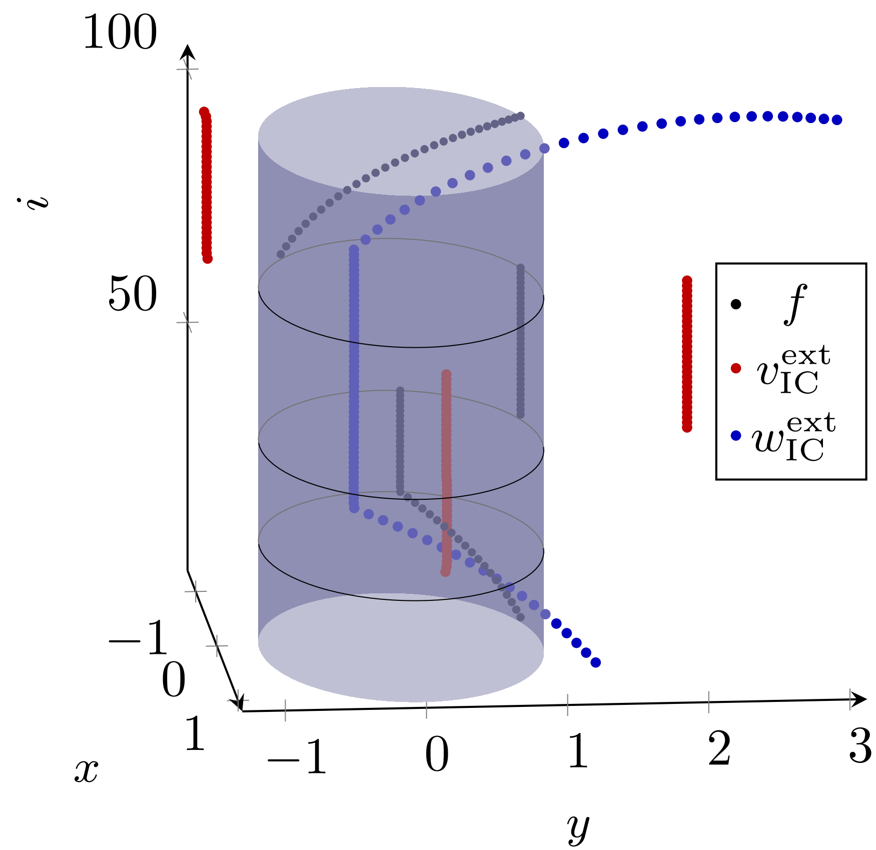
3. **Total Generalized Variation**:

$$J_{\mathrm{TGV}}^{\mathrm{ext}}(u) := \frac{1}{2}\|f - u\|_2^2 + \alpha \, \mathrm{TGV}(u) + \iota_{\mathcal{M}^N}(u)$$

- Numerical solution by ADMM; convergence results if $\mathcal{M}^N$ is a closed, convex set

Refs: Rosman/Tai/Kimmel/Bruckstein 2014 for $L_2^2 - \mathrm{TV}$





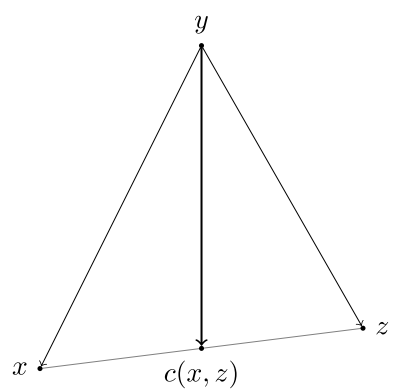
Decomposition of a piecewise geodesic signal f with values in $\mathbb{S}^1$ can be only depicted in the embedding space $\mathbb{R}^2$ of $\mathbb{S}^1$

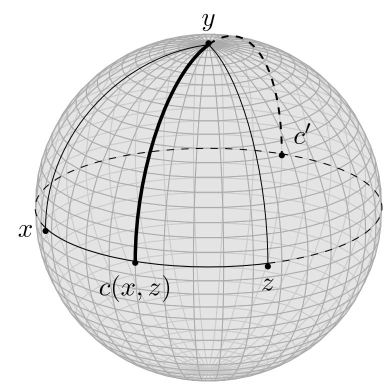
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40

Data Term: $\mathcal{D}(u;f) = \frac{1}{2}\sum_{i\in\mathcal{G}} \operatorname{dist}^2(f_i,u_i) = \frac{1}{2}\,\mathbf{dist}^2(f,u)$

TV Prior: $\operatorname{TV}(u) := \sum_{i\in\mathcal{G}}\Big(\sum_{j\in\mathcal{N}(i)} \operatorname{dist}^2(u_i,u_j)\Big)^{\frac{1}{2}}.$

Refs: Strekalovsky/Cremers 2011, Lellmann/Strekalovsky/Cremers 2013, Weinmann/Demaret/Storath 2014





Problem: Second Order Priors

Euclidean setting: $\mathrm{d}_2(x,y,z) = \|x-2y+z\|_2 = 2\|\underbrace{\frac{1}{2}(x+z)}_{\text{midpoint}}-y\|_2$

Manifold setting: $\mathrm{d}_2(x,y,z) := \min_{c\in\mathcal{C}_{x,z}} d(c,y)$ where

$$\mathcal{C}_{x,z} := \Big\{c\in\mathcal{M} : c = \gamma_{\widehat{x,z}}\big(\tfrac{T}{2}\big) \text{ for any geodesic } \gamma_{\widehat{x,z}}\Big\}$$

Intrinsic Midpoint Model for Additive and IC Model

Setting

$$\mathrm{TV}_2^{\mathrm{int}}(u) := \sum_{i \in \mathcal{G}} \Big((\mathrm{d}_{xx}^{\mathrm{int}} u)_i^2 + (\mathrm{d}_{yy}^{\mathrm{int}} u)_i^2 + (\mathrm{d}_{xy}^{\mathrm{int}} u)_i^2 + (\mathrm{d}_{xy}^{\mathrm{int}} u)_i^2 \Big)^{\frac{1}{2}}$$

where

$$(\mathrm{d}_{xx}^{\mathrm{int}} u)_i := \begin{cases} \mathrm{d}_2(u_{i+(1,0)}, u_i, u_{i-(1,0)}) & \text{if } i \pm (1,0) \in \mathcal{G}, \\ 0 & \text{otherwise} \end{cases}$$

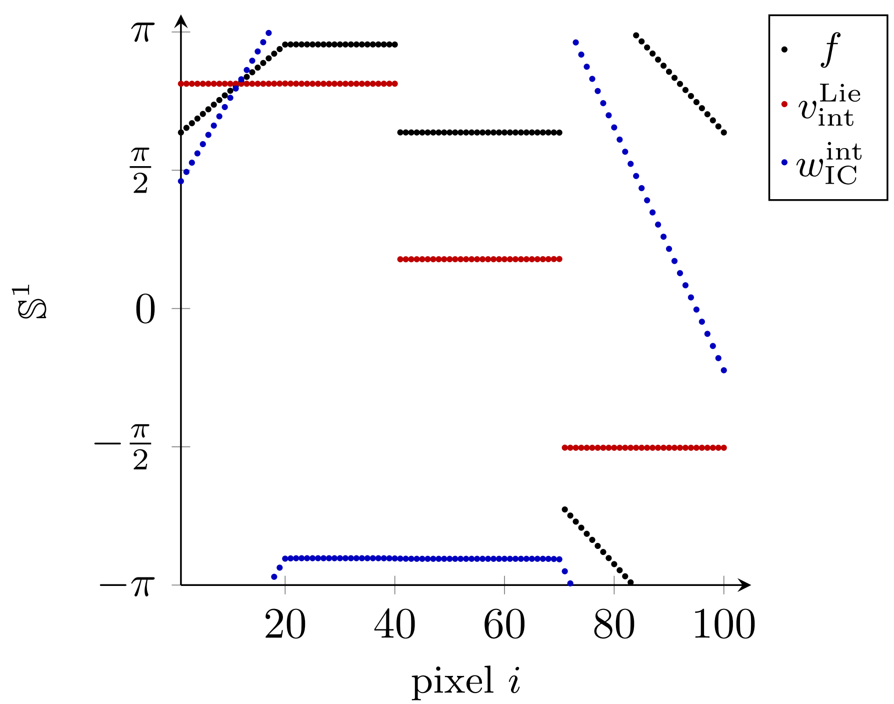
and similarly for $\mathrm{d}_{xx}^{\mathrm{int}} u$ and $\mathrm{d}_{xy}^{\mathrm{int}} u$ we obtain:

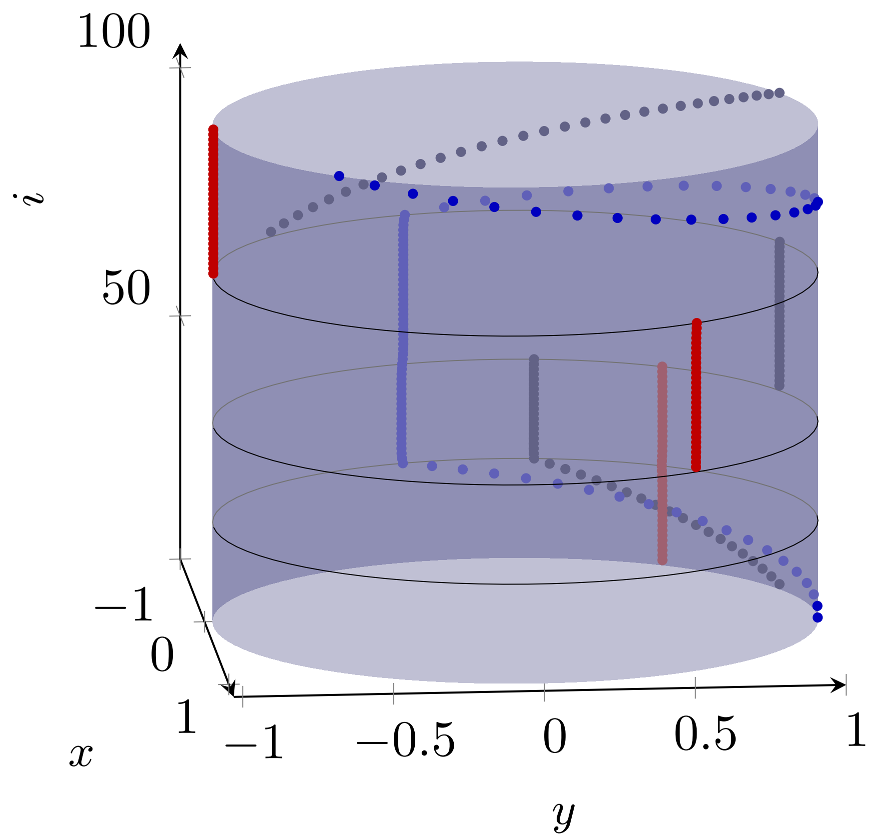
1. **Additive Coupling**

$$\mathrm{TV}_{1\wedge 2}^{\mathrm{int}}(u) := \beta \mathrm{TV}^{\mathrm{int}}(u) + (1-\beta)\mathrm{TV}_2^{\mathrm{int}}(u)$$

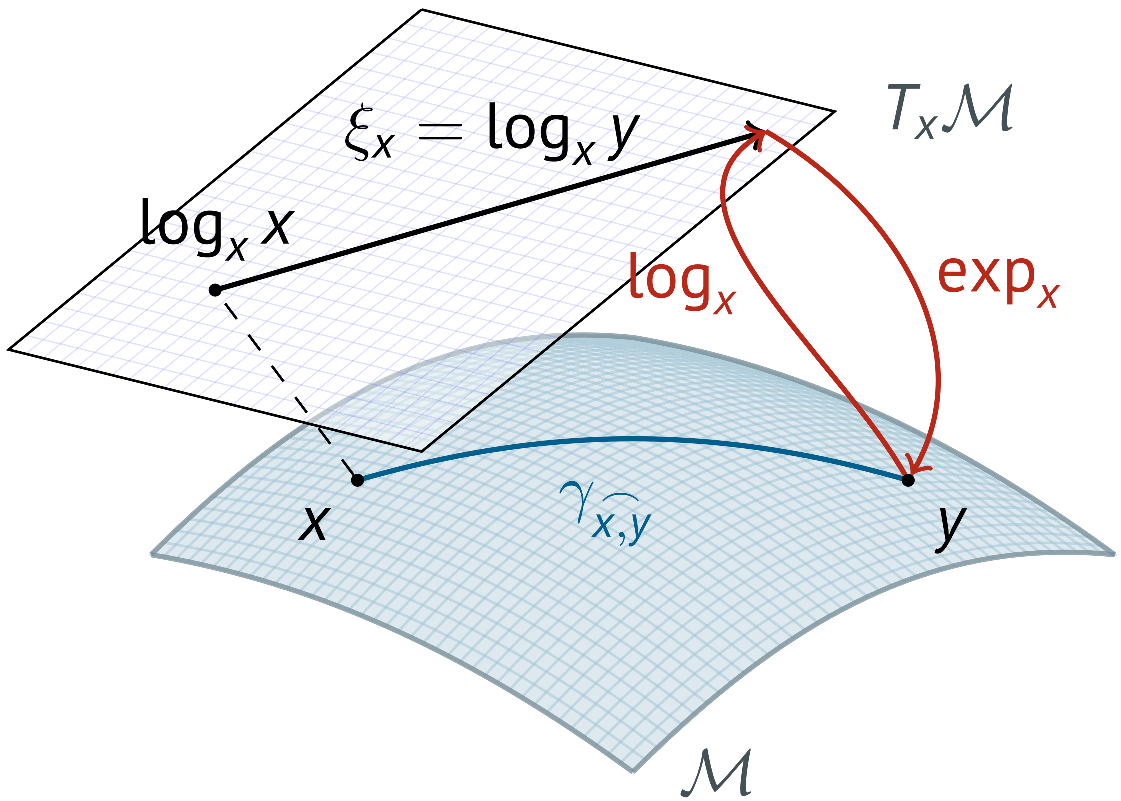
2. **Infimal Convolution**

$$\mathrm{IC}^{\mathrm{int}}(u) := \inf_{u \in \mathcal{C}_{v,w}} \{\beta \mathrm{TV}^{\mathrm{int}}(v) + (1-\beta)\mathrm{TV}_2^{\mathrm{int}}(w)\}.$$





Decomposition of a piecewise geodesic signal f with values in $\mathbb{S}^1$ by midpoint IC



$T_x\mathcal{M}$
$\xi_x = \log_x y$
$\log_x x$
$\log_x$
$\exp_x$
x
$\gamma_{\widehat{x,y}}$
y
$\mathcal{M}$

Euclidean Model:

$$\mathrm{TGV}(u) := \inf_a \left\{ \beta \int_\Omega |\nabla u - a| \,\mathrm{d}x + (1-\beta) \int_\Omega |\widetilde{\nabla} a| \,\mathrm{d}x \right\}$$

Manifold-Valued Model:

$$\nabla^{\mathrm{int}} := \begin{pmatrix} D_x^{\mathrm{int}} \\ D_y^{\mathrm{int}} \end{pmatrix}, \qquad \widetilde{\nabla}^{\mathrm{int}} := \begin{pmatrix} \widetilde{D}_x^{\mathrm{int}} & 0 \\ \widetilde{D}_y^{\mathrm{int}} & 0 \\ 0 & \widetilde{D}_x^{\mathrm{int}} \\ 0 & \widetilde{D}_y^{\mathrm{int}} \end{pmatrix}$$

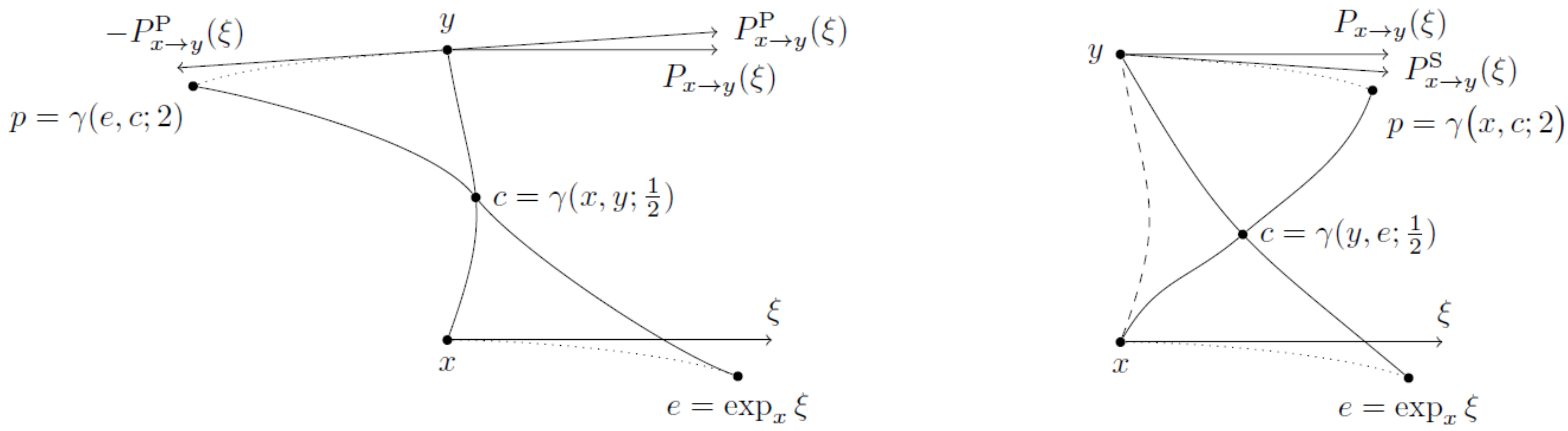
where

$$(D_x^{\mathrm{int}} u)_i := \begin{cases} \log_{u_i} u_{i+(1,0)} & \text{if } i+(1,0) \in \mathcal{G}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\widetilde{D}_x^{\mathrm{int}} \xi)_i := \begin{cases} a_i - P^{\mathrm{P}}_{u_{i-(1,0)} \to u_i}(a_{i-(1,0)}) & \text{if } i \pm (1,0) \in \mathcal{G} \\ 0 & \text{otherwise} \end{cases}$$

Intrinsic TGV Model



Pole ladder and Schild's ladder for approximating parallel transport

Refs: Pennec 2018: exactness of Pole ladder approximation for connected symmetric manifolds

3. Total Generalized Variation

$$\mathrm{TGV}^{\mathrm{int}}(u) := \inf_a \Big\{ \beta \|\nabla^{\mathrm{int}} u - a\|_{2,1,u} + (1-\beta)\|\widetilde{\nabla}^{\mathrm{int}} a\|_{2,1,u} \Big\}$$

Refs: Bredies/Holler/Storath/Weinmann 2018

Idea: Replace addition in differences by the group operation

1. **Additive Coupling**:

$$\mathcal{J}^{\mathrm{Lie}}_{\mathrm{Add}}(u) := \frac{1}{2}\mathbf{dist}^2(f,u) + \alpha\big(\beta\mathrm{TV}(u) + (1-\beta)\mathrm{TV}^{\mathrm{Lie}}_2(u).\big)$$

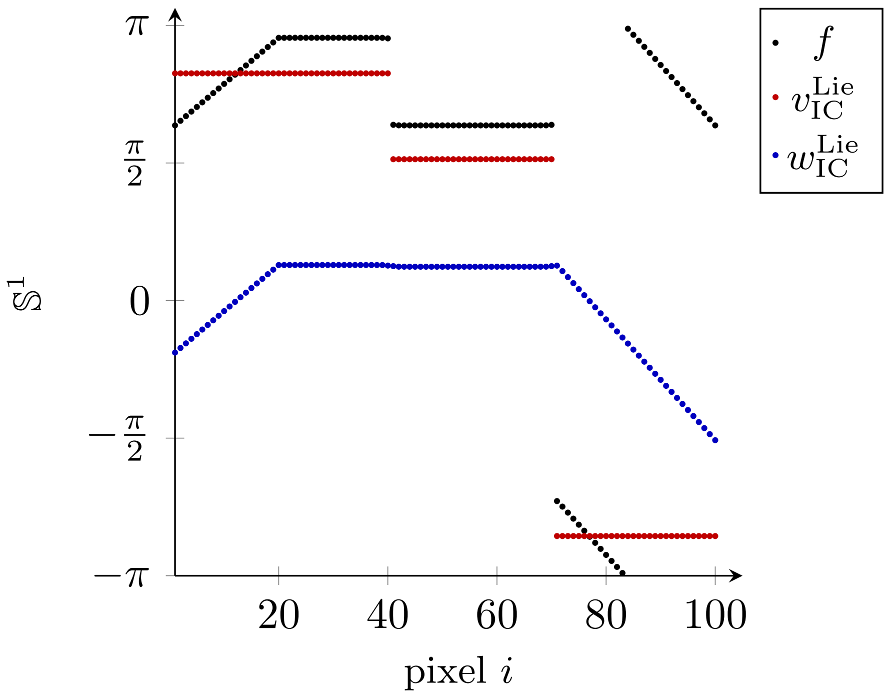
2. **Infimal Convolution**:

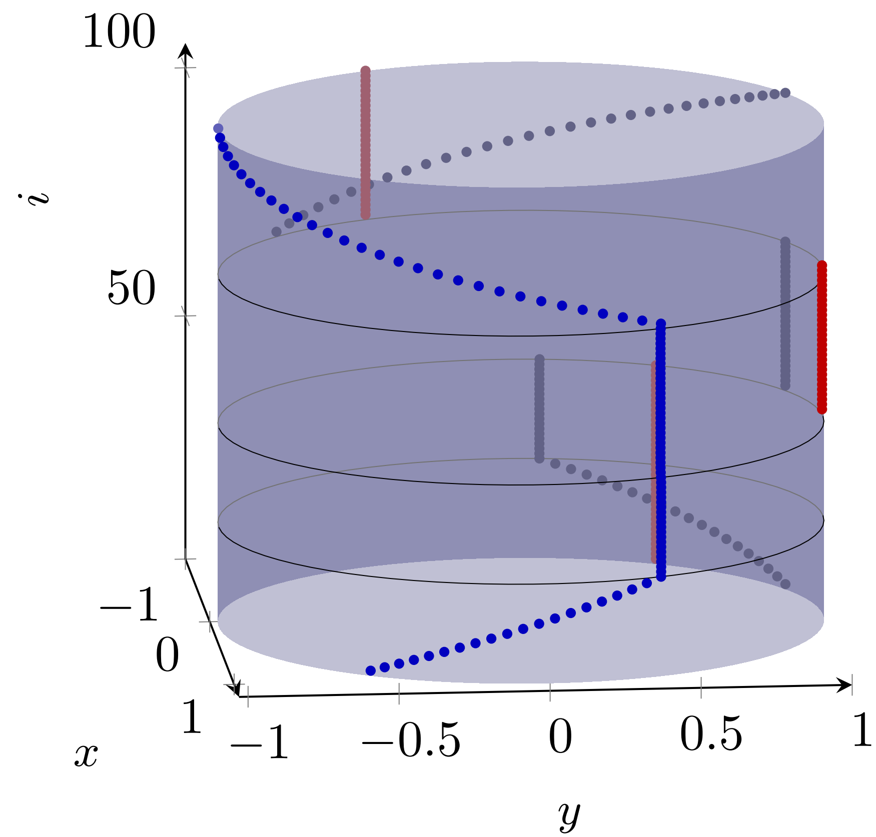
$$J^{\mathrm{Lie}}_{\mathrm{IC}}(u) := \frac{1}{2}\mathbf{dist}^2(f,u) + \alpha \inf_{u=v\circ w}\big(\beta\mathrm{TV}(v) + (1-\beta)\mathrm{TV}^{\mathrm{Lie}}_2(w)\big).$$

3. **Total Generalized Variation**:

$$J^{\mathrm{Lie}}_{\mathrm{TGV}}(u) := \frac{1}{2}\mathbf{dist}^2(f,u) + \alpha\inf_{a}\big(\beta\,\mathbf{dist}_{2,1}(\nabla^{\mathrm{Lie}}u,a) + (1-\beta)\,\mathbf{dist}_{2,1}(\widetilde{\nabla}^{\mathrm{Lie}}a,e)\big).$$

Example: Intrinsic IC Model via Lie Groups





Decomposition of a piecewise geodesic signal f with values in $\mathbb{S}^1$ by Lie group IC model

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Nearly all algorithms require

- Chain rule (up to five times)
- Gradient computations of $\exp_\cdot(y)$, $\exp_y(\cdot)$, $\log_\cdot(y)$, $\exp_y(\cdot)$ $\gamma_{\cdot,y}$, $\gamma_{y,\cdot}$ via Jacobi fields (ODE solution)

Further we need

(b) Cyclic Proximal Point Algorithm

- Proximal mappings

(c) Parallel Douglas Rachford Algorithm

- Reflections

(d) Half-Quadratic Regularization/Iteratively Reweighted Least Squares

- Alternating algorithm for generalized functional

Refs: Nikolova/Ng 2005, Nikolova/Chan 2007

Cyclic Proximal Point Alg. (CPPA) and Parallel Douglas-Rachford Alg. (DR) require multiple computations of certain proximal mappings.

For $\varphi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ the proximal mapping is defined by

$$\operatorname{prox}_{\lambda\varphi}(g) := \operatorname*{argmin}_{u \in \mathbb{R}^N} \{\frac{1}{2}\|u-g\|_2^2 + \lambda\varphi(u)\}$$

For $\varphi : \mathcal{M}^N \to \mathbb{R} \cup \{+\infty\}$ by

$$\operatorname{prox}_{\lambda\varphi}(g) := \operatorname*{argmin}_{u \in \mathcal{M}^N} \{\frac{1}{2}\sum_{j=1}^N d\big(u(j), g(j)\big)^2 + \lambda\varphi(u)\}$$

For proper, convex, lsc functions φ on Hadamard spaces as e.g., $\mathcal{H}^N$ the minimizer exits and is uniquely determined (Jost 1995, Mayer 1998).

$$\mathcal{E}(u) := \sum_{i \in \mathcal{V}} d\big(f(i), u(i)\big)^2 + \alpha_1 \sum_{i \in \mathcal{G}} d\big(u(i+1), u(i)\big) + \alpha_2 \sum_{i \in \mathcal{G}} \mathrm{d}_2\big(u(i+1), u(i), u(i-1)\big)$$

Algorithm (CPPA): Find $\operatorname{argmin}_x \sum_{j=1}^{L} \varphi_j(x)$ by

$$x^{(k)} = \operatorname{prox}_{\lambda_k \varphi_L} \circ \operatorname{prox}_{\lambda_k \varphi_{L-1}} \circ \ldots \circ \operatorname{prox}_{\lambda_k \varphi_1}(x^{(k-1)})$$

We need

- appropriate splitting
- proximal mappings for
  - $\varphi(u) = d\big(f(i), u(i)\big)^2$, analytical solution (Ferreira, Oliveira 2002)
  - $\varphi(u) = d\big(u(i+1), u(i)\big)$, analytical solution (Weinmann, Demaret, Storath 2014)
  - $\varphi(u) = \mathrm{d}_2\big(u(i+1), u(i), u(i-1)\big)$
    - analytical solution for $\mathbb{S}^1$ (Bergmann, Laus, St, Weinmann 2014)
    - numerical solution by using Jacobi fields of the variation of the geodesics (Bacak, Bergmann, St, Weinmann 2016)

Euclidean setting: $\hat{x} \in \operatorname{argmin}_{x \in \mathbb{R}^m} \sum_{k=1}^{L} \varphi_k(x)$

Setting

$$\Phi(\mathbf{x}) := \sum_{k=1}^{n} \varphi_k(x_k),$$

$$\mathsf{D} := \left\{\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^{Lm} \colon x_1 = \ldots = x_n \in \mathbb{R}^m\right\} \subset \mathbb{R}^{Lm}$$

we obtain the minimization problem

$$\operatorname*{argmin}_{\mathbf{x} \in \mathbb{R}^{Lm}} \left\{\Phi(\mathbf{x}) + \iota_{\mathsf{D}}(\mathbf{x})\right\}$$

Parallel DR Algorithm:

$$\mathbf{t}^{(r+1)} := \left((1-\lambda_r)\,\mathrm{Id} + \lambda_r \underbrace{\mathcal{R}_{\eta\Phi}\mathcal{R}_{\iota_{\mathsf{D}}}}_{\text{reflections}}\right)\left(\mathbf{t}^{(r)}\right)$$

$$\hat{x} := \operatorname{prox}_{\eta\iota_D}(\hat{t}) = \frac{1}{L}\sum_{k=1}^{L} \hat{t}_k \qquad \text{final step}$$

References: Combettes, Pesquet 2008, 2014, Borwein, Tam 2008

Manifold setting (symmetric spaces): $\hat{x} \in \operatorname{argmin}_{x \in \mathcal{M}^m} \sum_{k=1}^{L} \varphi_k(x)$

Setting

$$\begin{aligned} \Phi(\mathbf{x}) &:= \sum_{k=1}^{n} \varphi_k(x_k), \\ \mathsf{D} &:= \left\{ \mathbf{x} \in \mathcal{M}^{Lm} : x_1 = \ldots = x_n \in \mathcal{M}^m \right\} \subset \mathcal{M}^{Lm} \end{aligned}$$

we obtain the minimization problem

$$\operatorname*{argmin}_{\mathbf{x} \in \mathcal{M}^{Lm}} \left\{ \Phi(\mathbf{x}) + \iota_{\mathsf{D}}(\mathbf{x}) \right\}$$

Parallel DR Algorithm:

$$\begin{aligned} \mathbf{s}^{(r)} &= \mathcal{R}_{\eta\Phi} \mathcal{R}_{\iota_{\mathsf{D}}} (\mathbf{t}^{(r)}) \\ \mathbf{t}^{(r+1)} &= \gamma_{\overset{\frown}{\mathbf{t}^{(r)}, \mathbf{s}^{(r)}}} (\lambda_r) \\ \hat{x} &= \mathbf{1}_n \otimes \operatorname*{argmin}_{x \in \mathcal{M}^m} \left\{ \sum_{k=1}^{n} d^2(x_k, x) \right\} \quad \text{Karcher mean} \end{aligned}$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40

$$\mathcal{E}(u) := \frac{1}{2}\sum_{i\in\mathcal{V}} d^2\big(f(i),u(i)\big) + \alpha\sum_{i\in\mathcal{G}}\sum_{j\in\mathcal{N}(i)^+} \varphi\big(d\,(u(i),u(j))\,\big)$$

- φ differentiable and some other properties, example: $\varphi(x) = \sqrt{x^2+\varepsilon^2}$

$$\mathcal{J}(u,v) := \frac{1}{2}\sum_{i\in\mathcal{V}} d^2\big(u(i),f(i)\big) + \alpha\sum_{i\in\mathcal{G}}\sum_{j\in\mathcal{N}(i)^+} \big(c\,(d(u(i),u(j)),v(i,j)) - \psi(v(i,j))\big)$$

with c-transform

$$\psi(s) := \inf_{t\in\mathbb{R}}\{c(t,s)-\varphi(t)\}$$

and

$$c(t,s) := t^2 s, \qquad \text{(multiplicative)}$$

$$c(t,s) := \tfrac{1}{2}\left(\sqrt{a}t - \frac{1}{\sqrt{a}}s\right)^2, \quad a>0, \quad \text{(additive)}$$

Alternating Algorithm:

$$v^{(k+1)} \in \underset{v}{\operatorname{argmin}}\, \mathcal{J}_\nu(u^{(k)}, v)$$

$$u^{(k+1)} \in \underset{u}{\operatorname{argmin}}\, \mathcal{J}_\nu(u, v^{(k+1)})$$

- **Iteratively Reweighted Least Squares**
  = Special Case of Half-Quadratic Minimization
  of multiplicative type with $\varphi(x) = \sqrt{x^2 + \varepsilon^2}$

References:

- Nikolova, Ng, Chan 2005, 2007
- Bergmann, Chan , Hielscher, Persch, St. 2016

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Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, where we assume $d\mathbb{P}(\omega) = p_X(x)\,dx$

- **Task**: Estimate $X : \Omega \to \mathbb{R}^p$ given $Y : \Omega \to \mathbb{R}^p$, i.e. we seek an estimator $T : \mathbb{R}^p \to \mathbb{R}^p$ such that $\hat{X} = T(Y)$ approximates X
- **Minimum Mean-Square Estimator (MMSE)** (Lehmann-Scheffe Thm)

$$T_{\text{MMSE}} \in \operatorname*{argmin}_T \mathbb{E}\|X - T(Y)\|_2^2 = \mathbb{E}(X|Y)$$

For

$$Y := X + \eta, \qquad X \sim \mathcal{N}(\mu_X, \Sigma_X), \quad \eta \sim \mathcal{N}(0, \sigma^2 I_p),$$

where X, η independent we get

$$T_{\mathsf{MMSE}}(Y) = \mu_Y + (\Sigma_Y - \sigma^2 I_p)\Sigma_Y^{-1}(Y - \mu_Y)$$

- **MMSE estimation for similar patches:** Choose $s \times s$ neighborhood (patch) y_i centered at $i = (i_1, i_2) \in \mathcal{G}$ and interpret this and similar patches as realization of an $p = s^2$-dimensional random vector $Y_i \sim \mathcal{N}(\mu_i, \Sigma_i)$

$$\hat{y}_j = \hat{\mu}_i + (\hat{\Sigma}_i - \sigma^2 I_p)\hat{\Sigma}_i^{-1}(y_j - \hat{\mu}_i), \qquad j \in \mathcal{S}(i).$$

where $\mathcal{S}(i)$ is the set of centers of patchs similar to y_i

- **First Problem:** What is Gaussian noise on manifolds?
- **MMSE estimation** for similar patches for manifold-valued images:

$$\hat{y}_j = \exp_{\hat{\mu}_i}\left((\hat{\Sigma}_i - \sigma^2 I_{pd})\hat{\Sigma}_i^{-1}(\log_{\hat{\mu}_i} y_j)\right), \qquad j \in \mathcal{S}(i)$$

Further issues:

- similarity measure for manifold-valued patches
- boundary patches
- flat areas
- second step: oracle image
- acceleration
- aggregation

Refs: Lebrun/Buardes/Morel 2013, Delon, Almansa ...

Matlab implementaion using

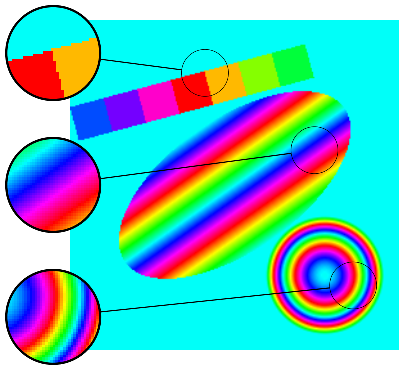
Manifold-Valued Image Restoration Toolbox (MVIRT)

by Bergmann and Persch

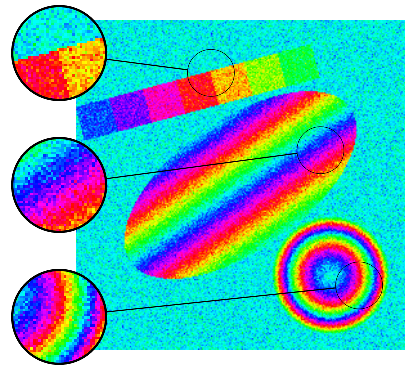
Performance measure:

$$\varepsilon = \frac{1}{N} \sum_{i \in \mathcal{G}} d_M^2(\hat{x}_i, x_i),$$

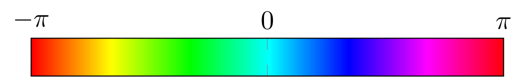
$\mathbb{S}^1$-valued Data

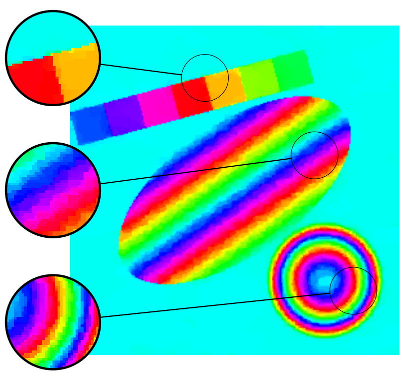


Original

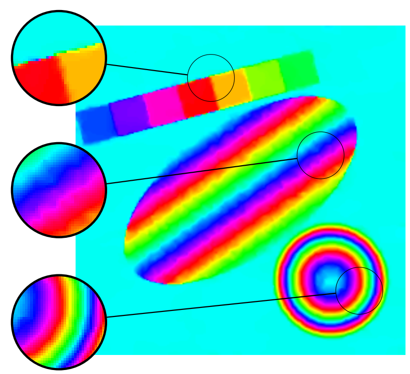


Noisy (88.5×10^{-3})

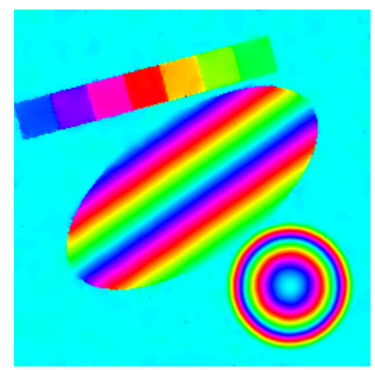




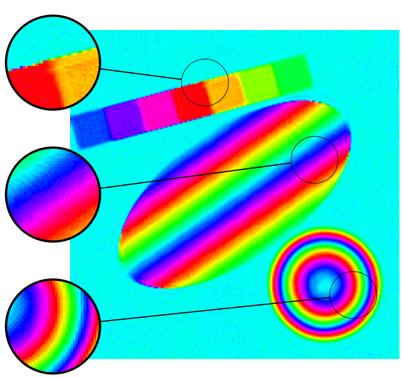
TV (7.2×10^{-3})



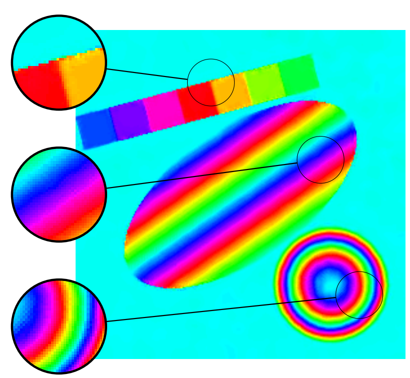
TV-TV2 (5.2×10^{-3})



TGV (2.6×10^{-3})

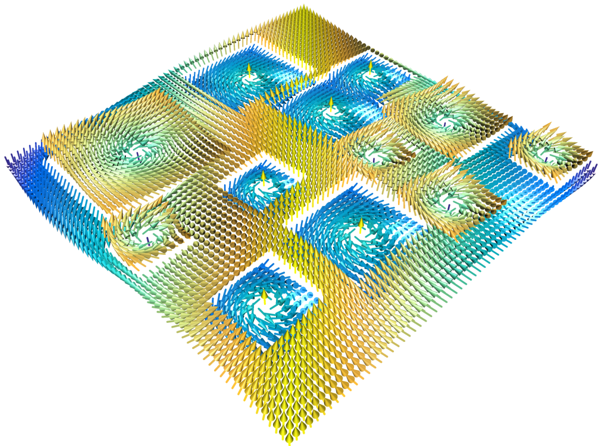


NL-means (8.1×10^{-3})

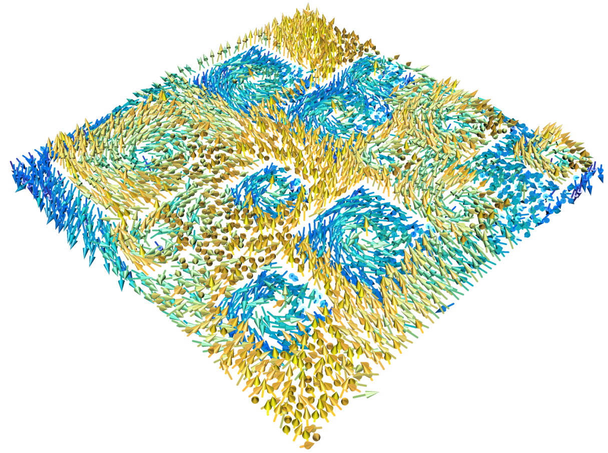


NL-MMSE (2.5×10^{-3})

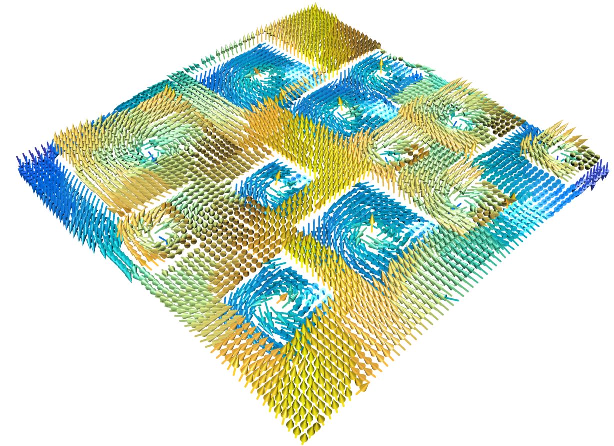
$\mathbb{S}^2$-valued Data



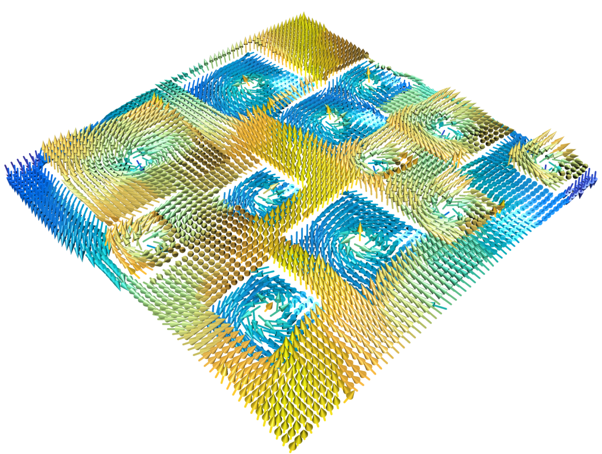
Original image



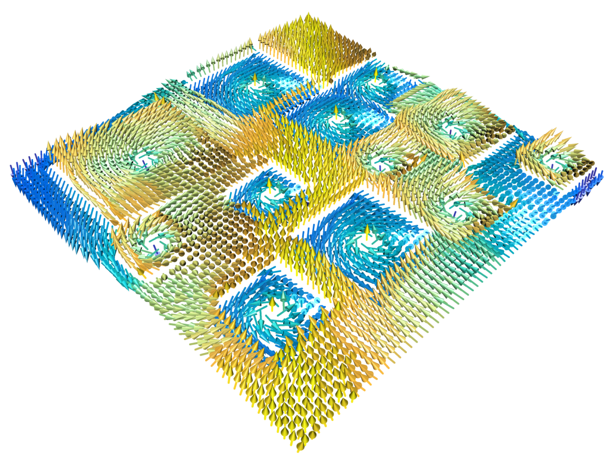
noisy image (0.1767)



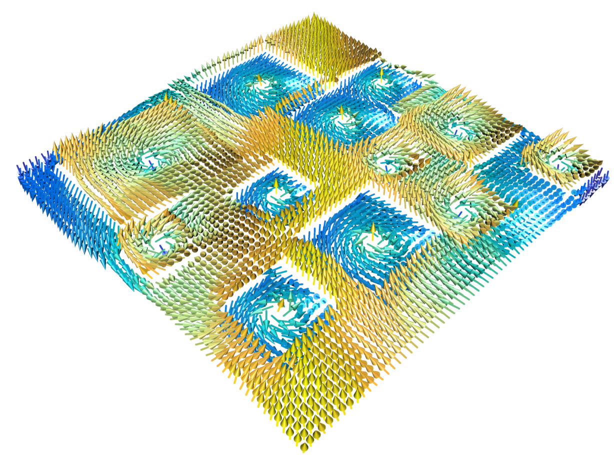
TV approach, (0.0352)



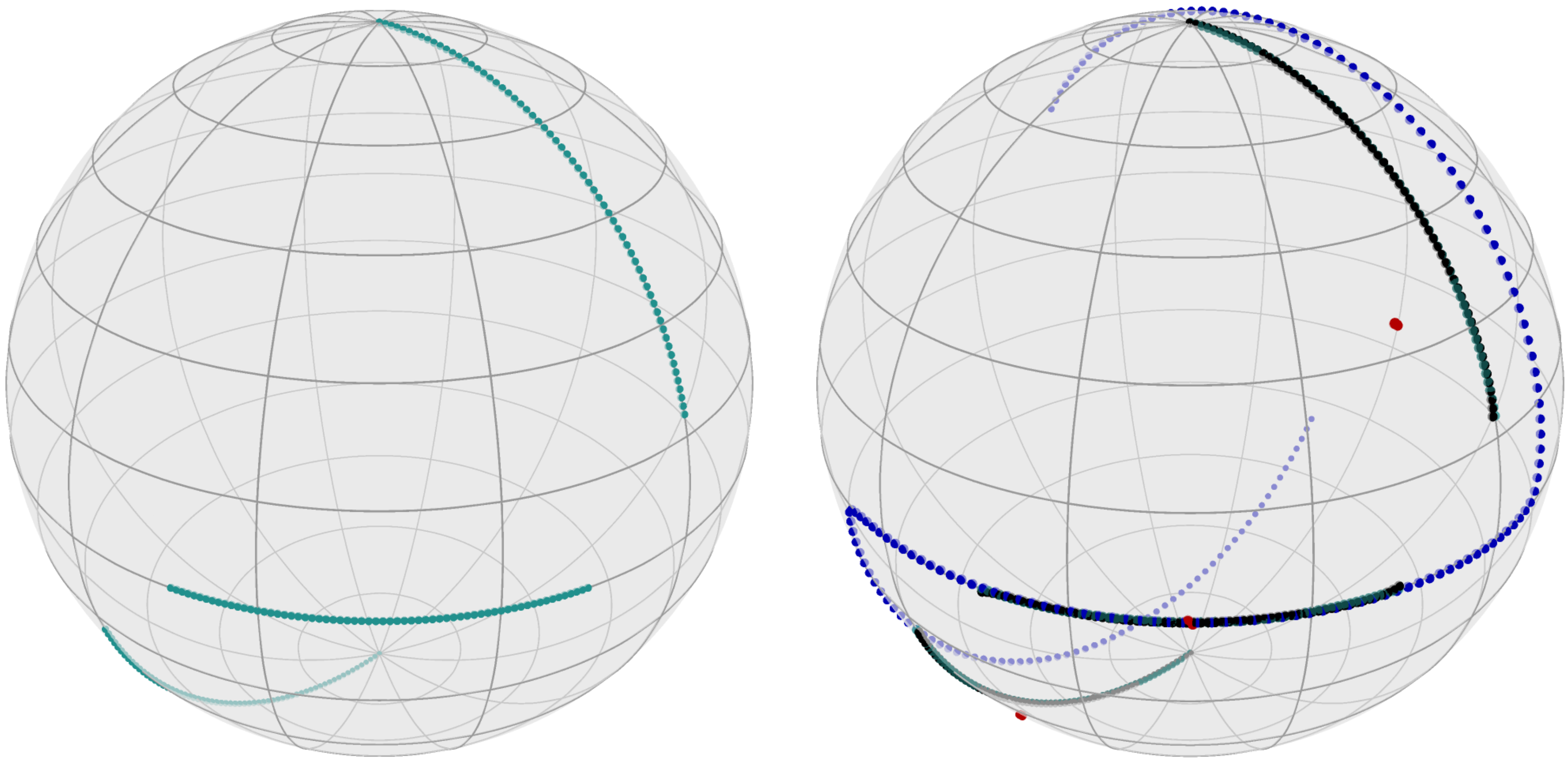
TV-TV_2 approach (0.0338)



NL-means $(\epsilon = 0.0326)$

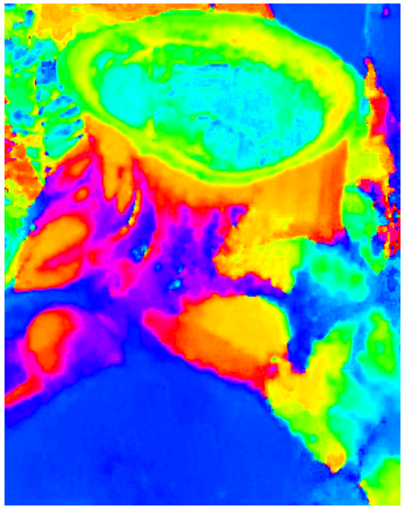


NL-MMSE $(\epsilon = 0.0258)$

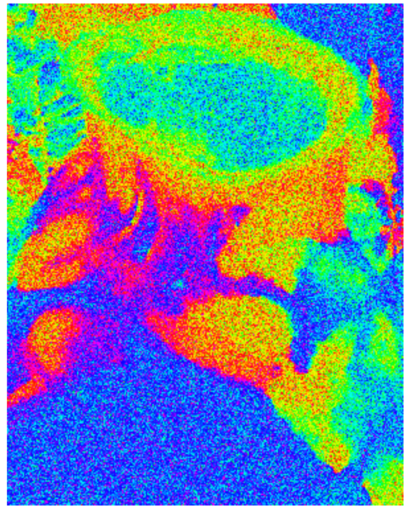


Midpoint IC decomposition of a piecewise geodesic signal f (green), into a piecewise constant part v (red) and a continuous piecewise geodesic curve w (blue). The mid point signal $u = \gamma_{\widehat{v,w}}(\frac{1}{2})$ (black) nearly reconstructs f (green)

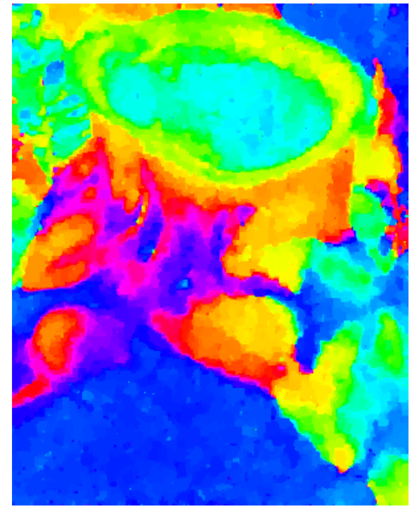
Color Channels



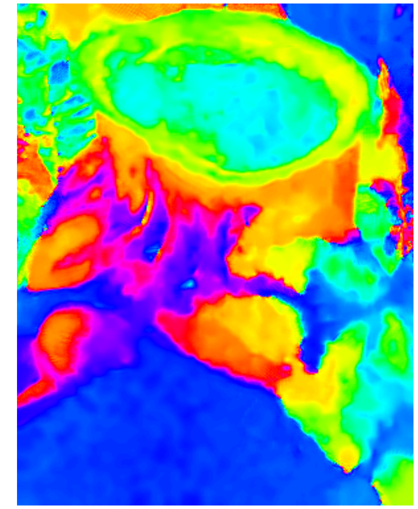
original hue



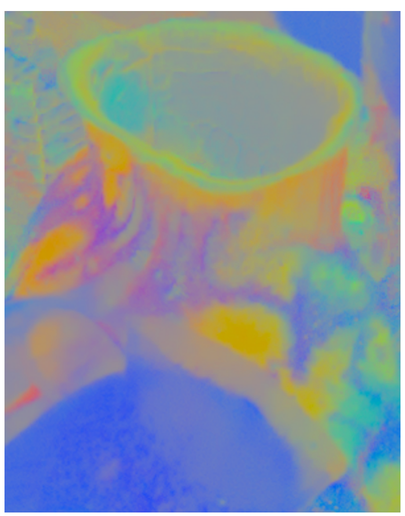
noisy hue (0.3609)



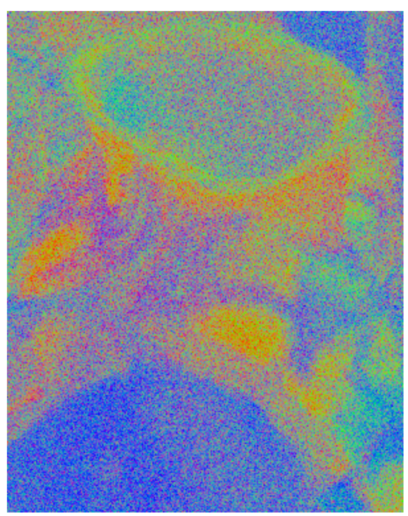
TV (0.0263)



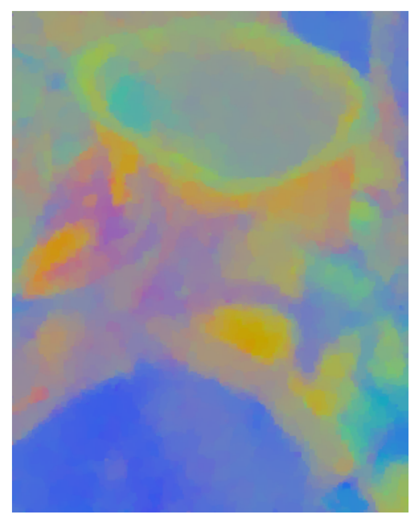
NL-MMSE (0.0194)



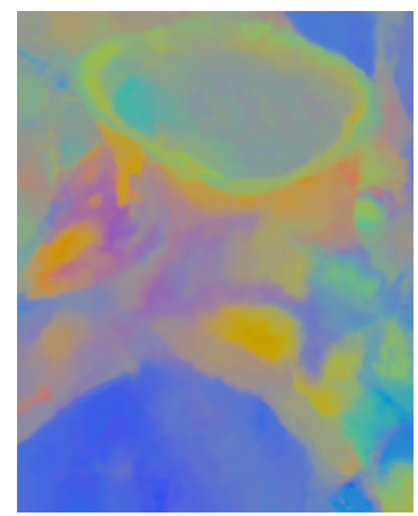
original chromaticity



Noisy chromaticity (0.0798)

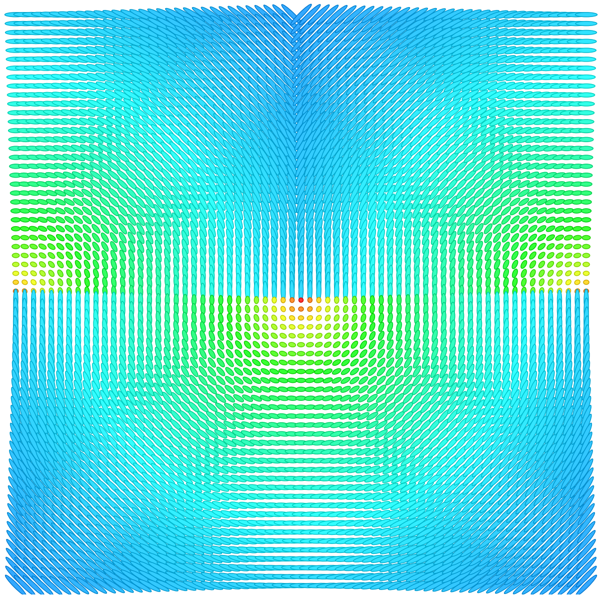


TV (0.0021)

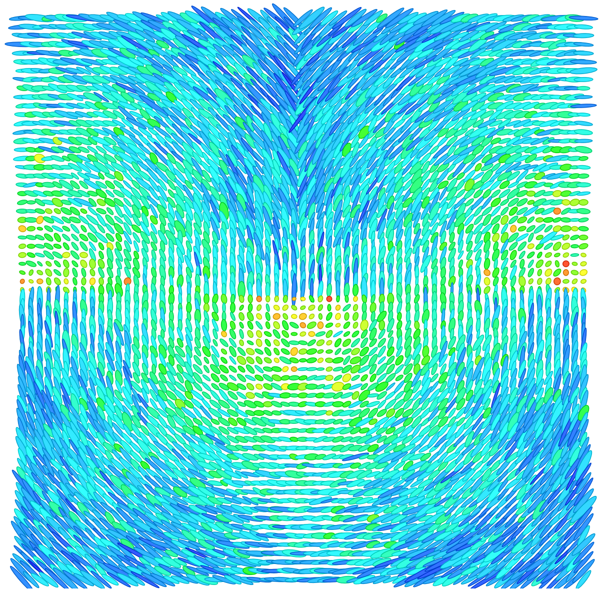


NL-MMSE (0.0017)

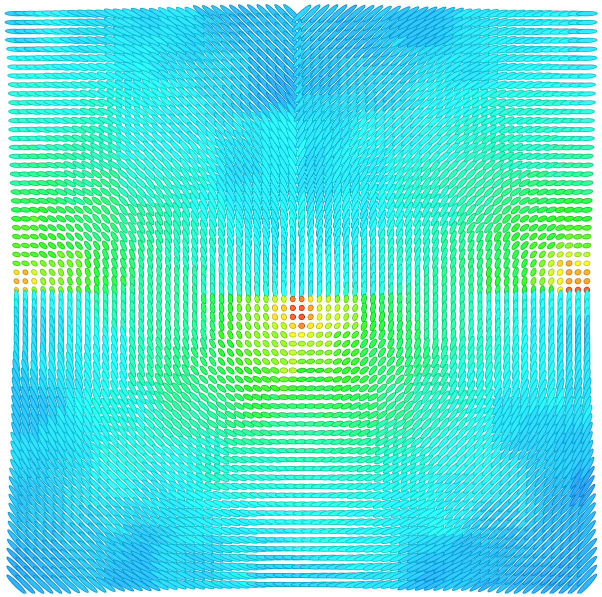
SPD(2) Data



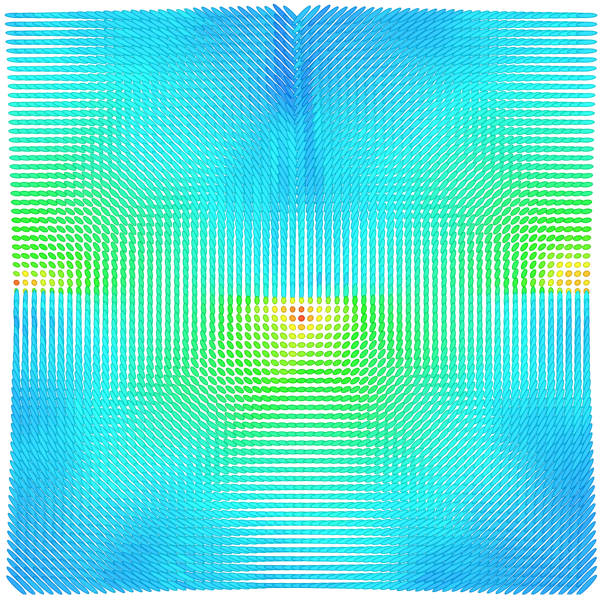
original image,



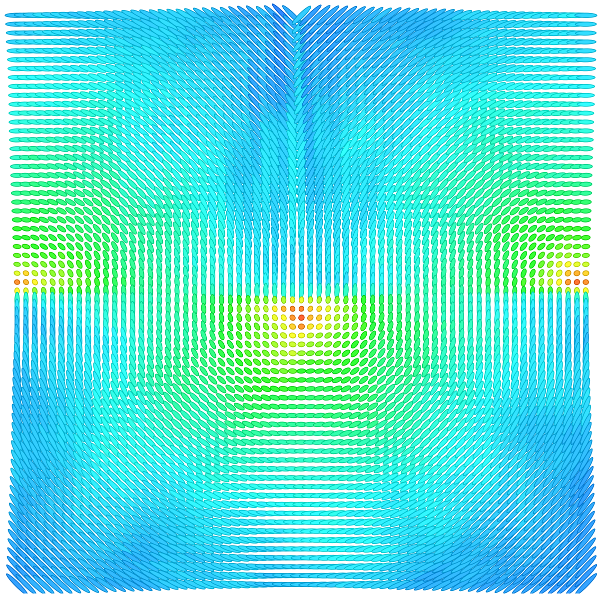
noisy image (0.1202)



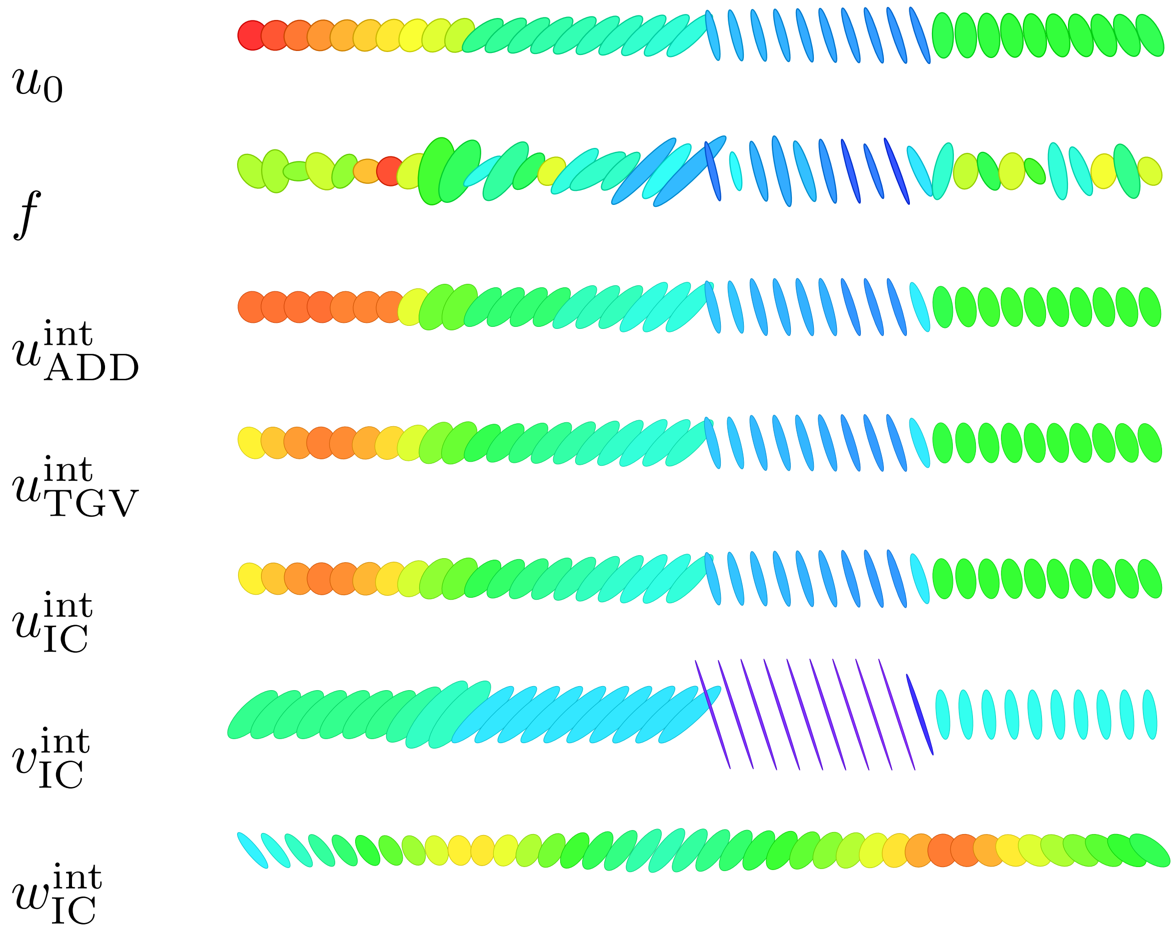
TV (0.0095)



NL-means (0.006)

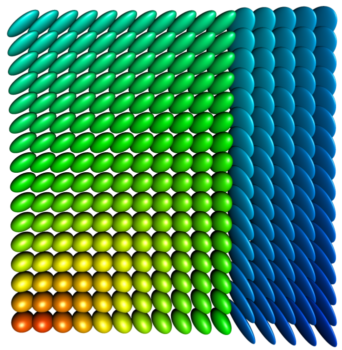


NL-MMSE (0.0043)

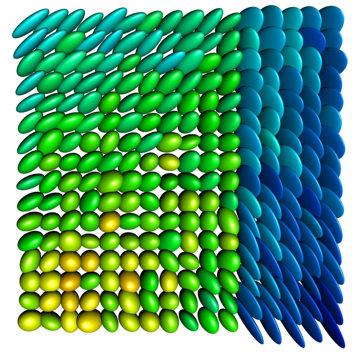


Denoising and decomposition of a $\mathcal{P}(2)$ valued signal f. Denoising result $u^{\text{int}}_{\text{ADD}}$ by additive model with prior $\mathrm{TV}^{\text{int}}_{1\wedge 2}$, $\epsilon = 0.0316$, with the TGV model, $\epsilon = 0.0259$, and $u^{\text{int}}_{\text{IC}}$ by Midpoint IC model, $\epsilon = 0.0269$. Decomposition by Midpoint IC model gives $v^{\text{int}}_{\text{IC}}$ and geodesic part $w^{\text{int}}_{\text{IC}}$ with geodesic midpoint $u^{\text{int}}_{\text{IC}}$.

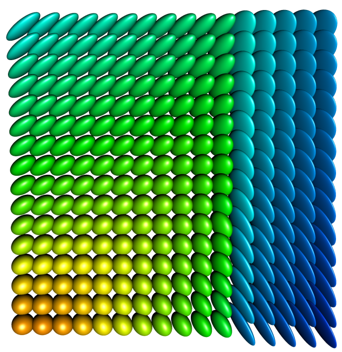
SPD(3) Data



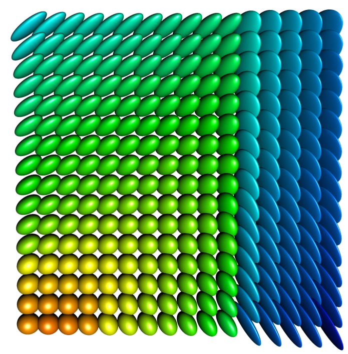
Orignal



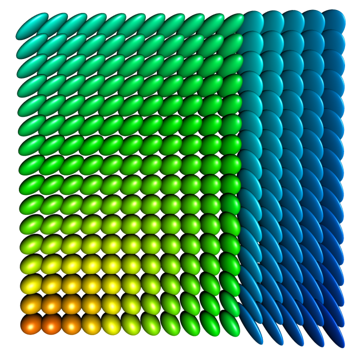
Noisy $\epsilon = 0.0583$



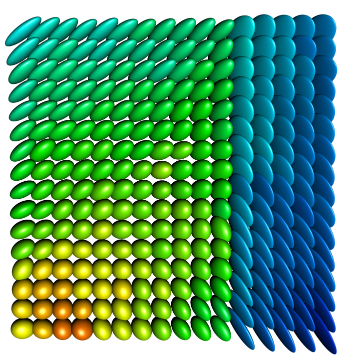
$u_{\mathrm{IC}}^{\mathrm{ext}}$, $\epsilon = 0.0066$



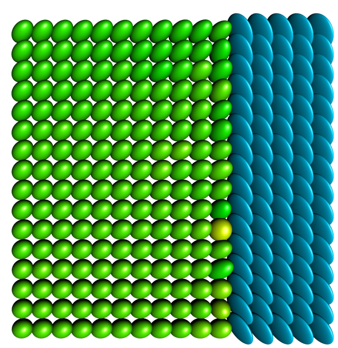
$u_{\mathrm{TGV}}^{\mathrm{ext}}$, $\epsilon = 0.0065$



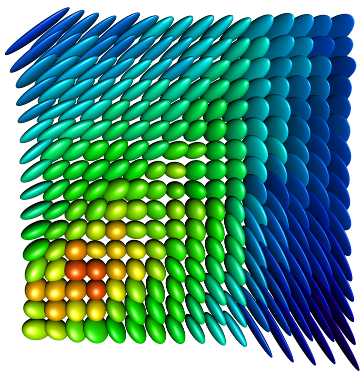
$u_{\mathrm{TGV}}^{\mathrm{int}}$, $\epsilon = 0.0034$



$u_{\mathrm{IC}}^{\mathrm{int}}$, $\epsilon = 0.0125$



$v_{\mathrm{IC}}^{\mathrm{int}}$



$w_{\mathrm{IC}}^{\mathrm{int}}$

THANK YOU FOR YOUR ATTENTION!

1. R. Bergmann, F. Laus, G. Steidl, and A. Weinmann. Second Order Differences of Cyclic Data and Applications in Variational Denoising. SIAM Imaging Sciences 2014.
2. R. Bergmann, R. H. Chan, R. Hielscher, J. Persch, G. Steidl. Restoration of Manifold-Valued Images by Half-Quadratic Minimization. Inverse Problems in Imaging 2016.
3. M. Bačák, R. Bergmann, G. Steidl and A. Weinmann. A Second Order Non-Smooth Variational Model for Restoring Manifold-Valued Images. SIAM Journal on Scientific Computing 2016.
4. R. Bergmann, J. Persch and G. Steidl. A Parallel Douglas Rachford Algorithm for Restoring Images with Values in Symmetric Hadamard Manifolds. SIAM on Imaging Sciences 2016.
5. R. Bergmann and A. Weinmann. A Second Order TV-Type Approach for Inpainting and Denoising higher dimensional cyclic and vector space data. Journal of Mathematical Imaging and Vision 2016.
6. R. Bergmann, J. H. Fitschen, J. Persch and G. Steidl. Iterative Multiplicative Filters for Data Labeling. International Journal of Computer Vision 2017.
7. F. Laus, **M. Nikolova**, J. Persch and G. Steidl. A Nonlocal Denoising Algorithm for Manifold-Valued Images Using Second Order Statistics. SIAM on Imaging Sciences 2017.
8. M. Bačák, M. Montag and G. Steidl. Convergence of Functions and their Moreau-Yosida Envelopes on Hadamard Spaces. Journal of Approximation Theory 2017
9. R. Bergmann, J. H. Fitschen, J. Persch and G. Steidl. Prior with Coupled First and Second Order Differences for Manifold-Values Image Processing. Journal of Mathematical Imaging and Vision 2018.

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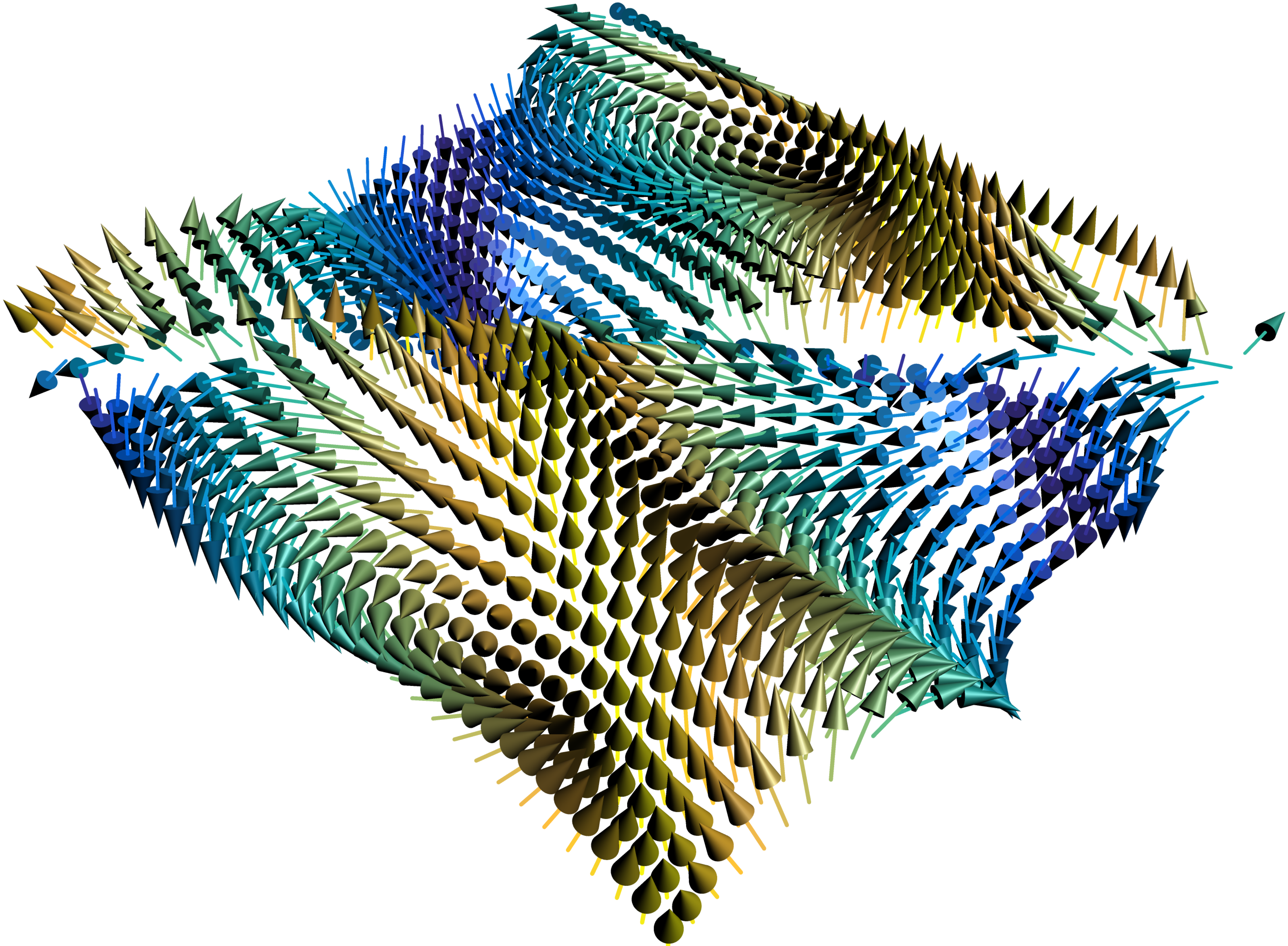

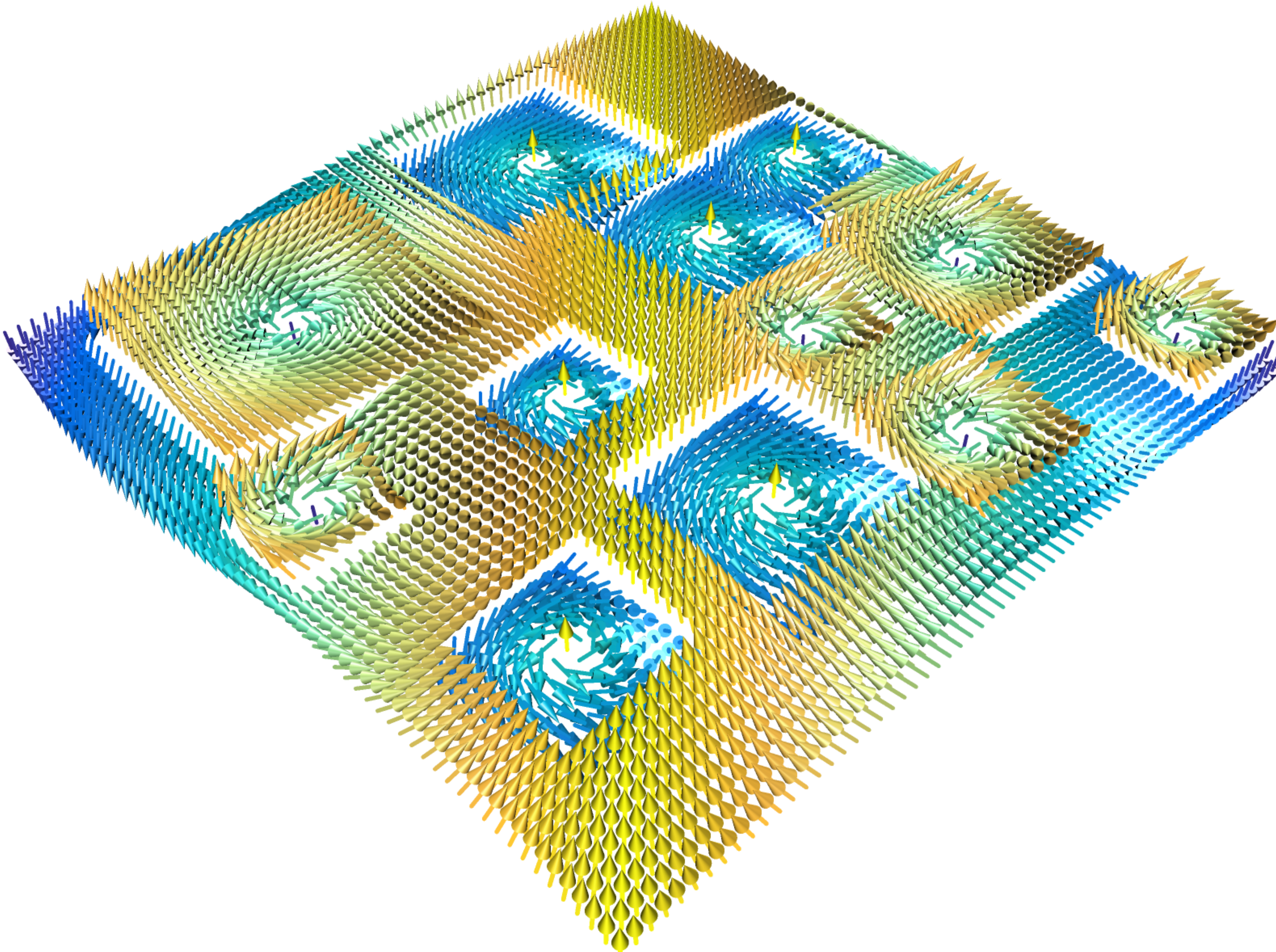

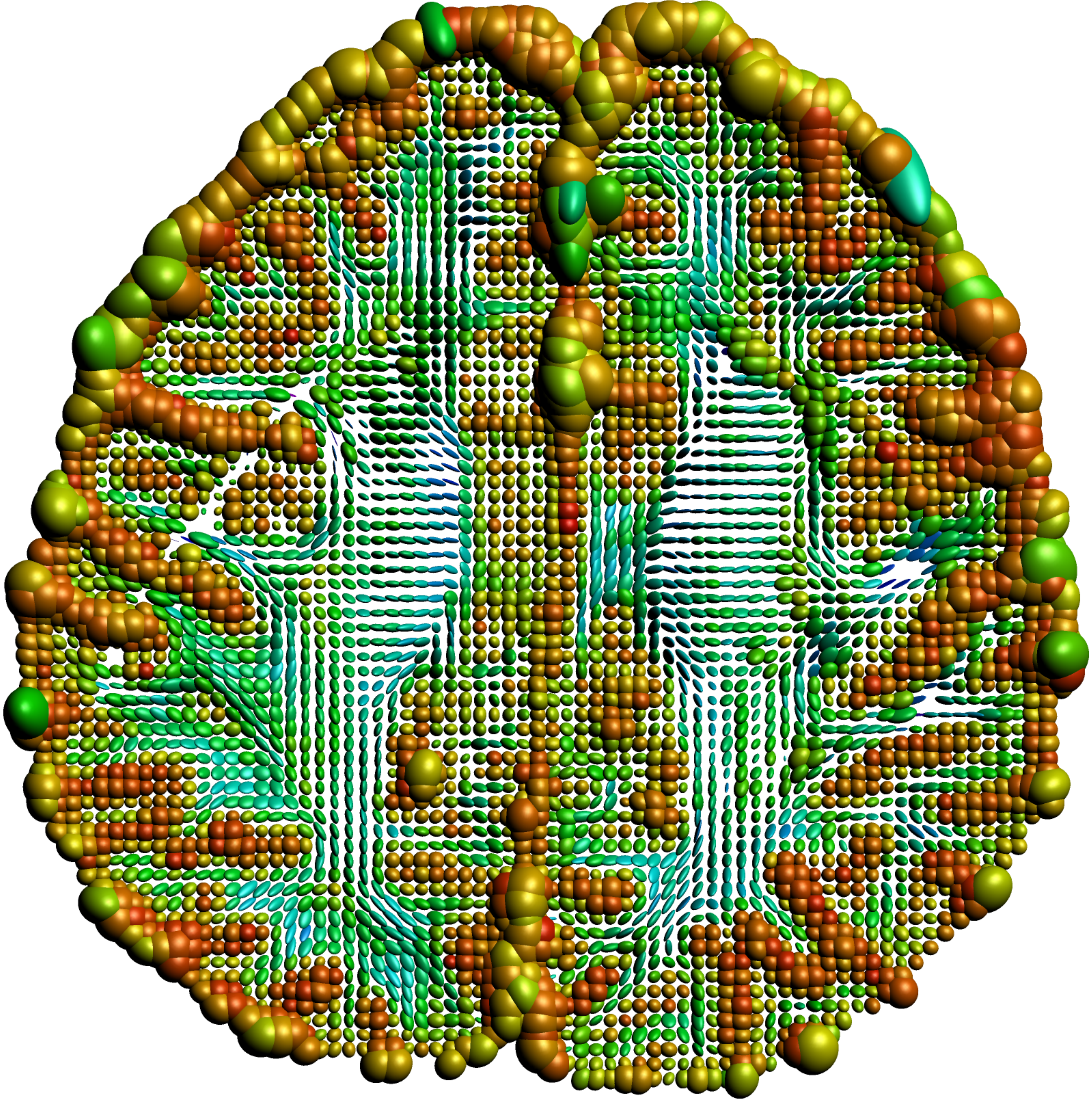

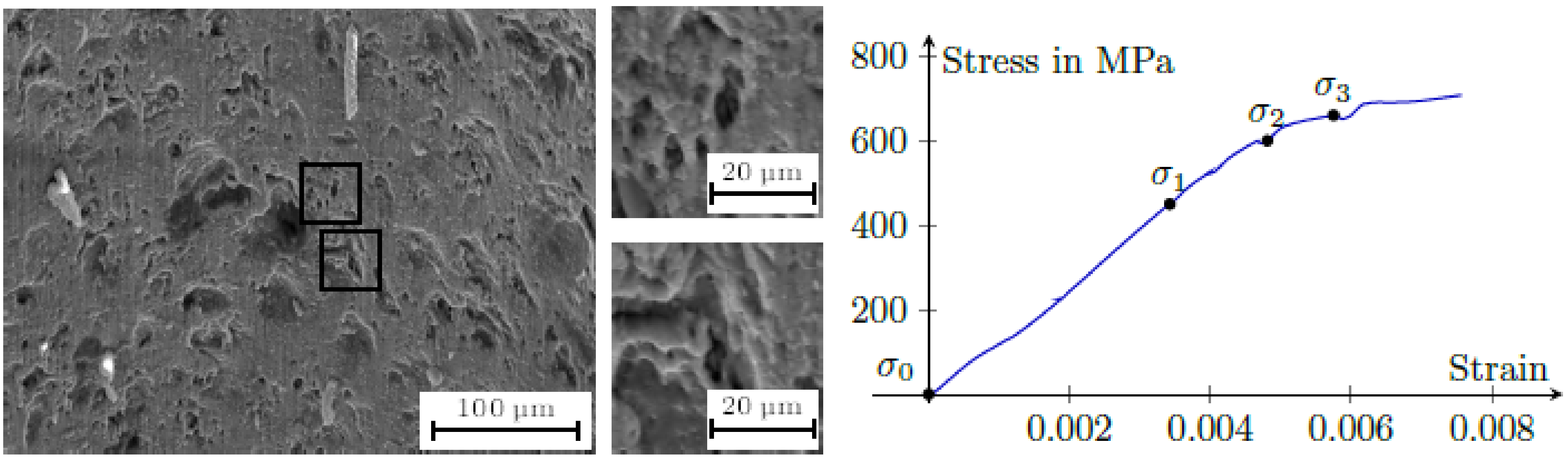
100 µm
20 µm
20 µm
Stress in MPa
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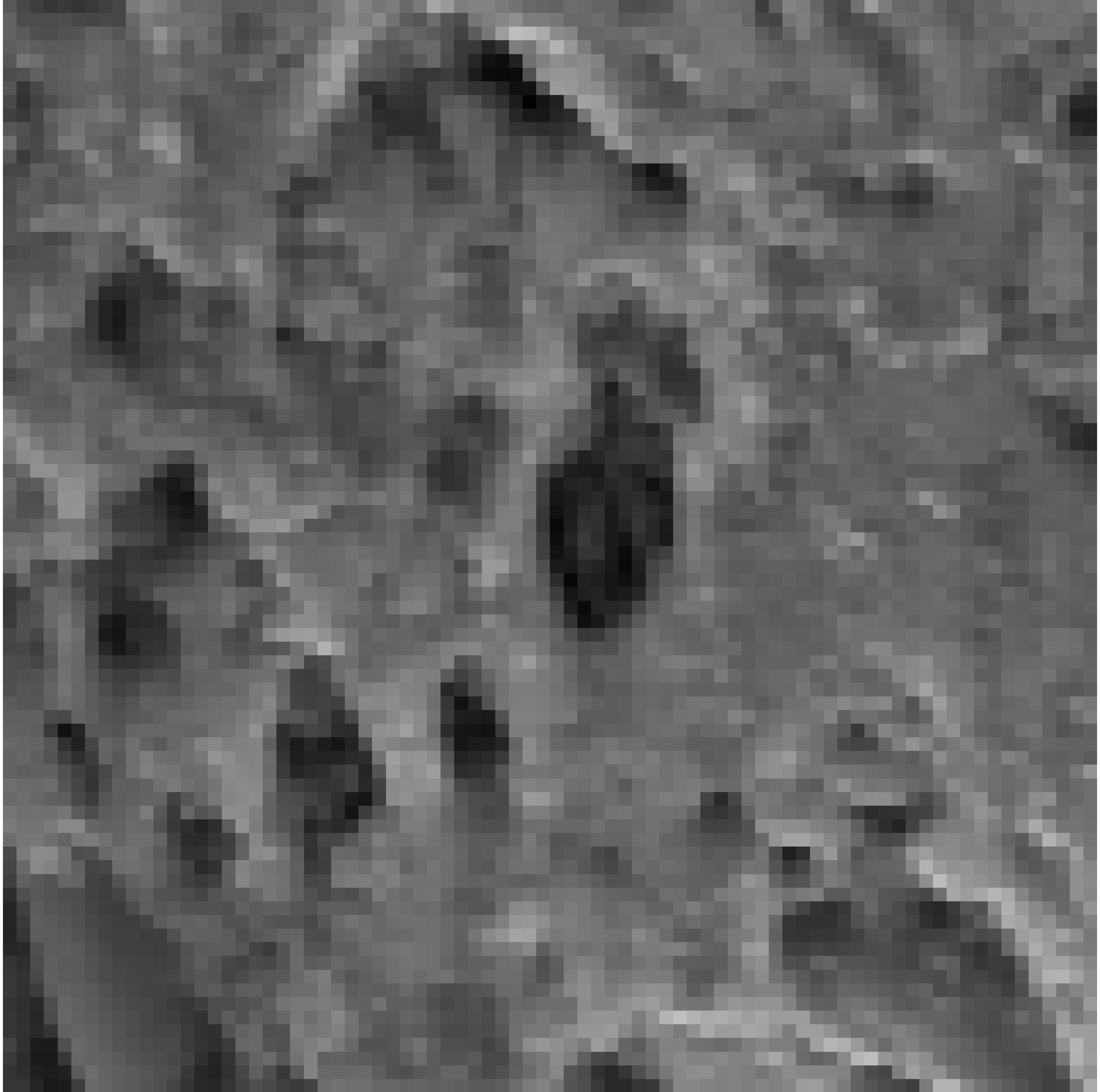

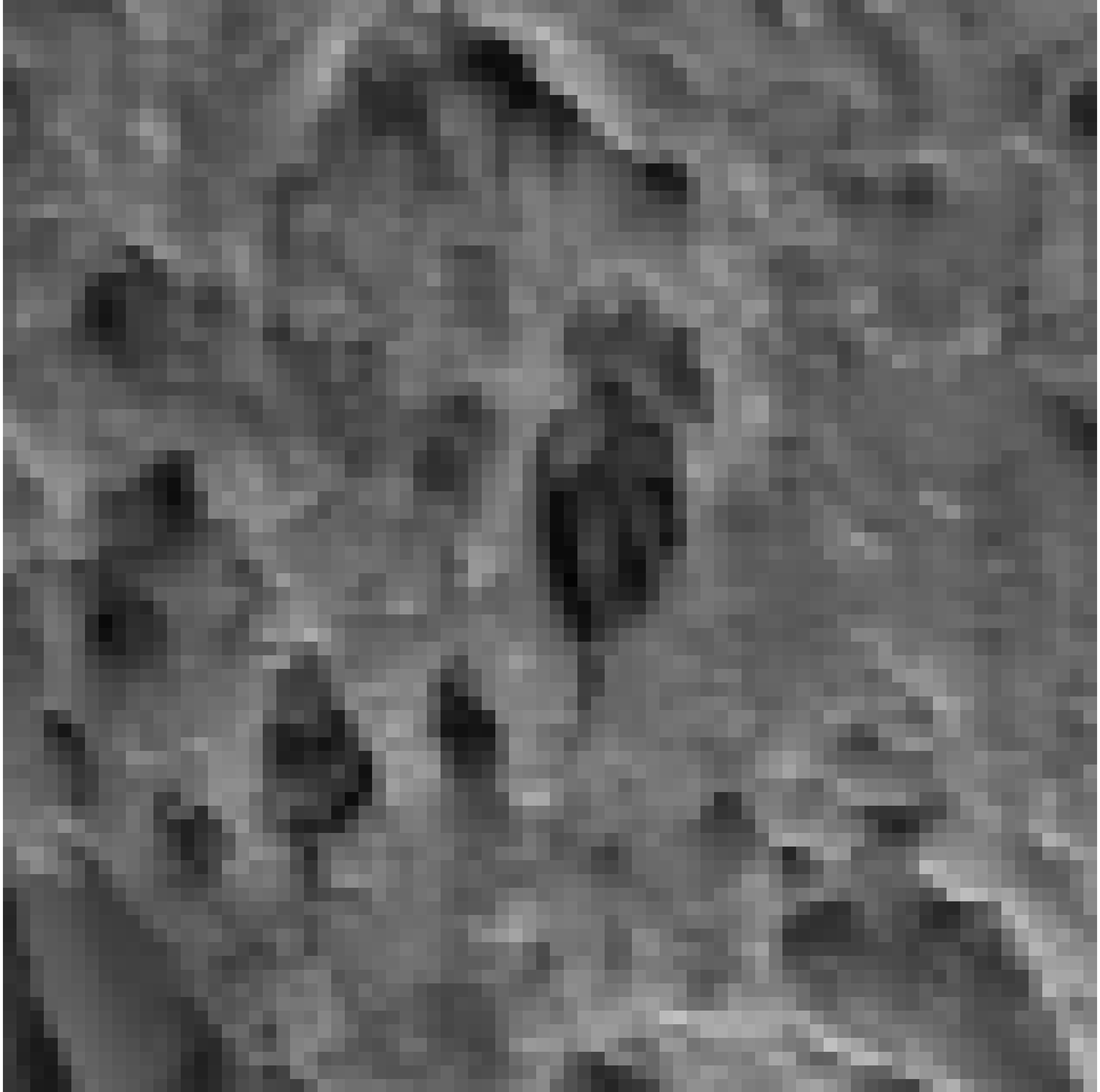

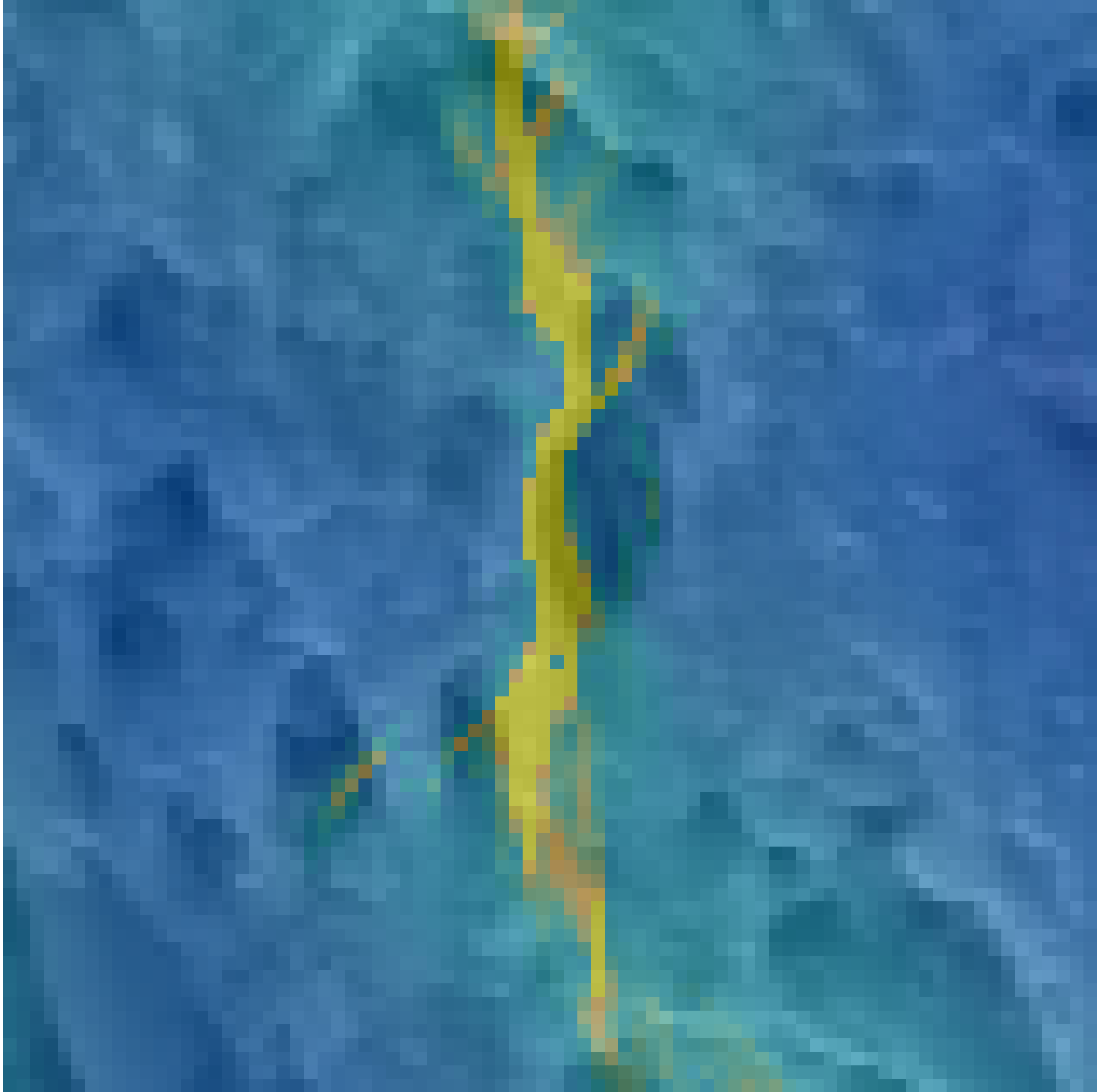

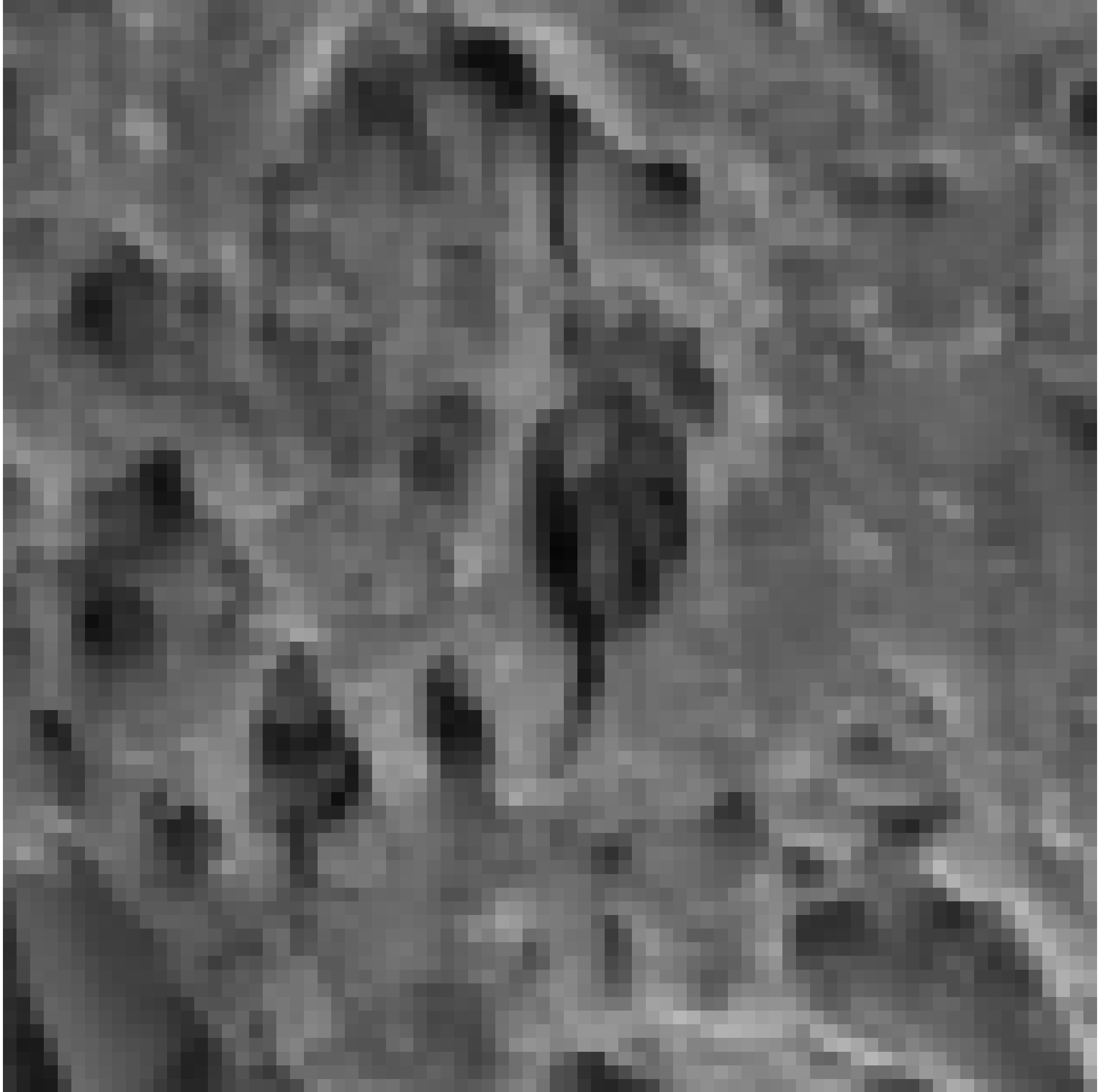

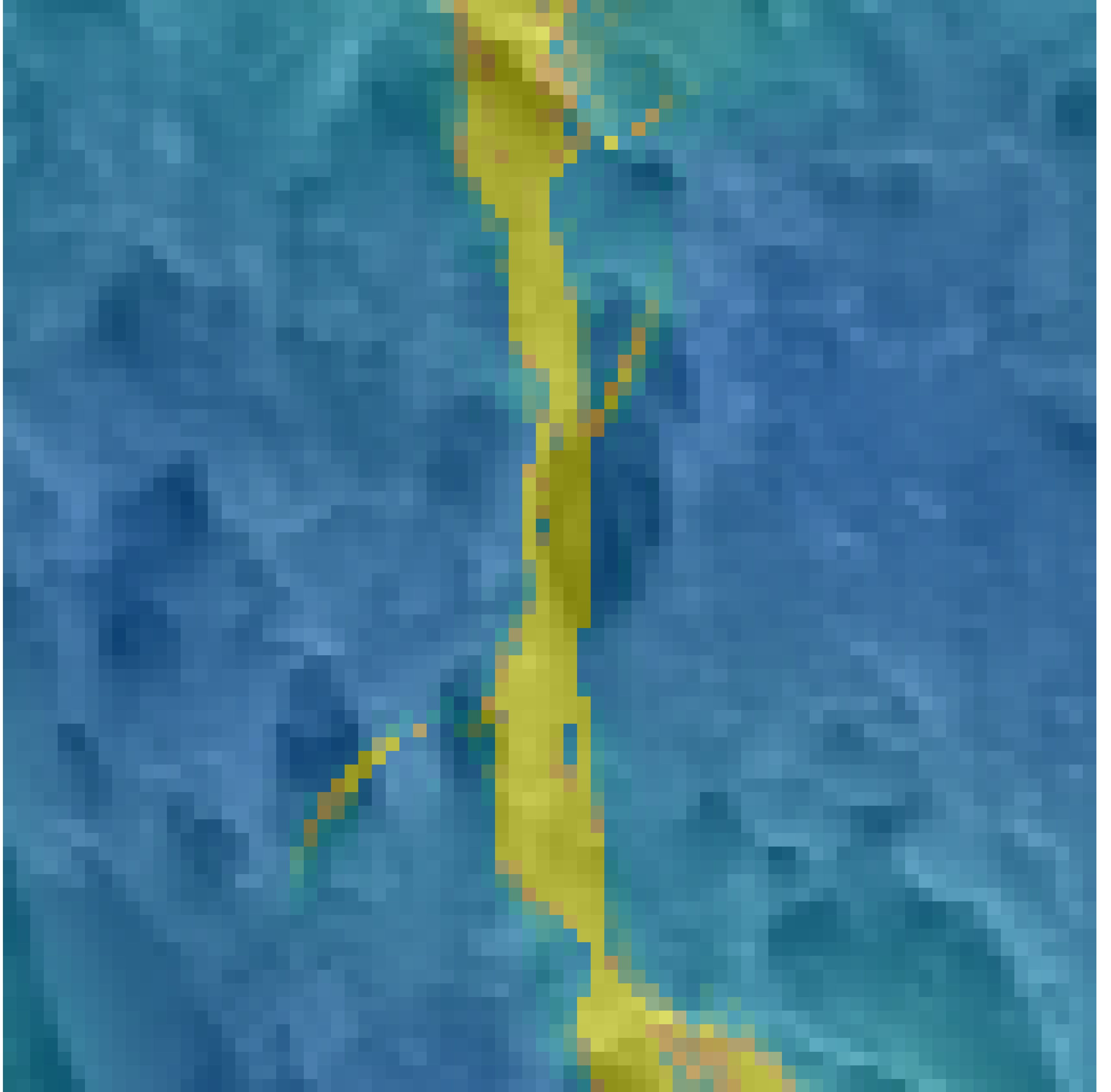

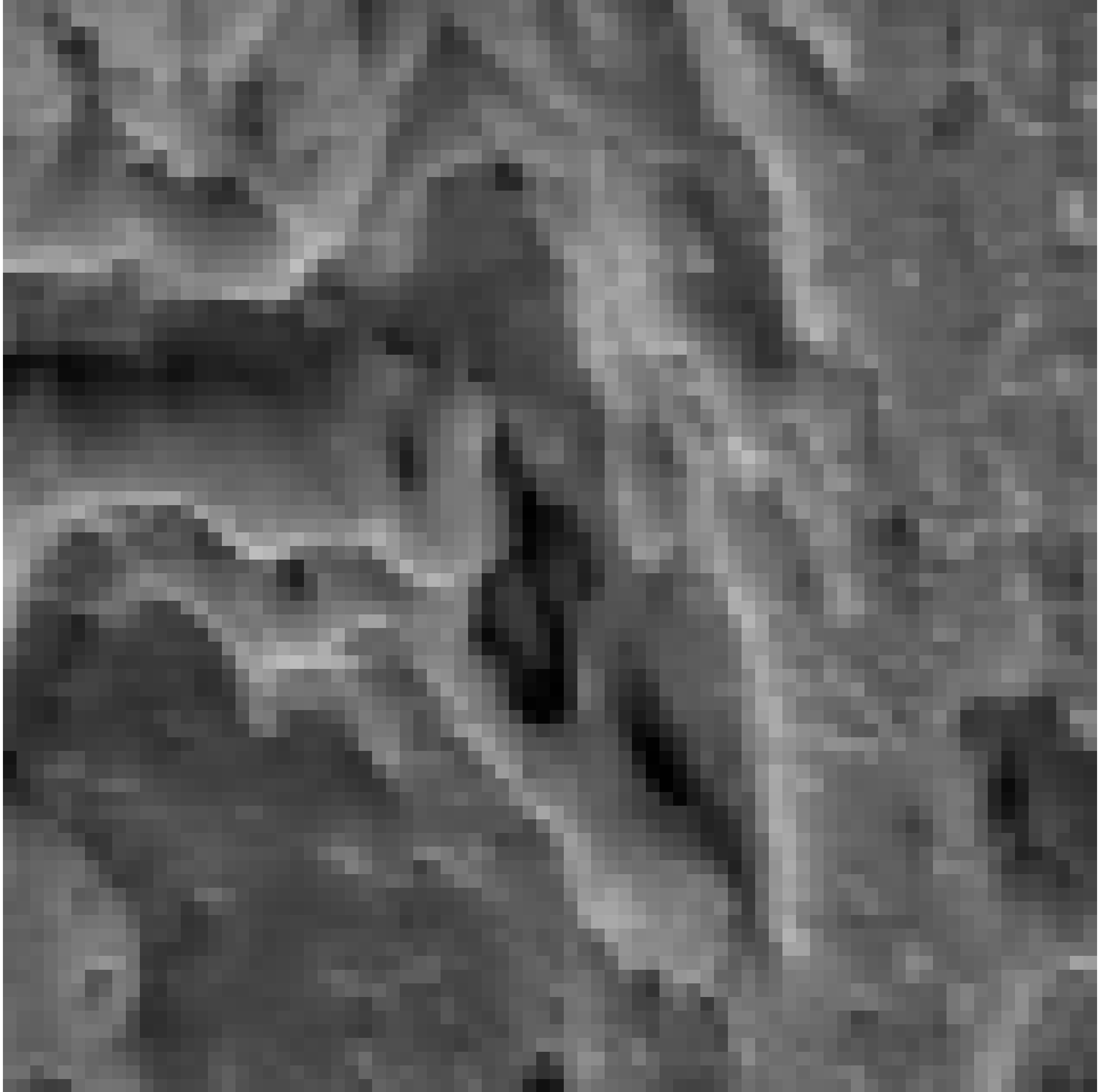

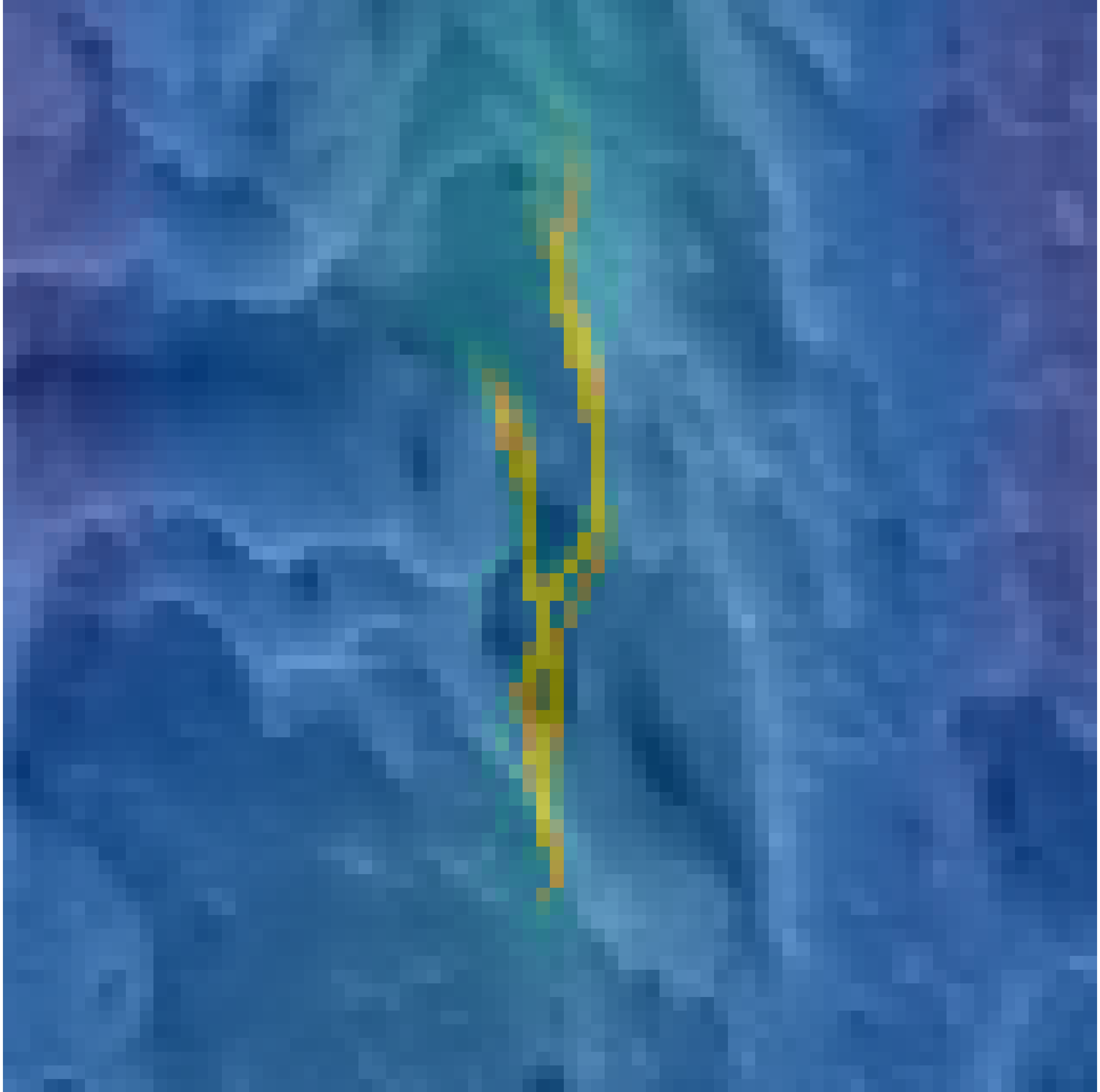

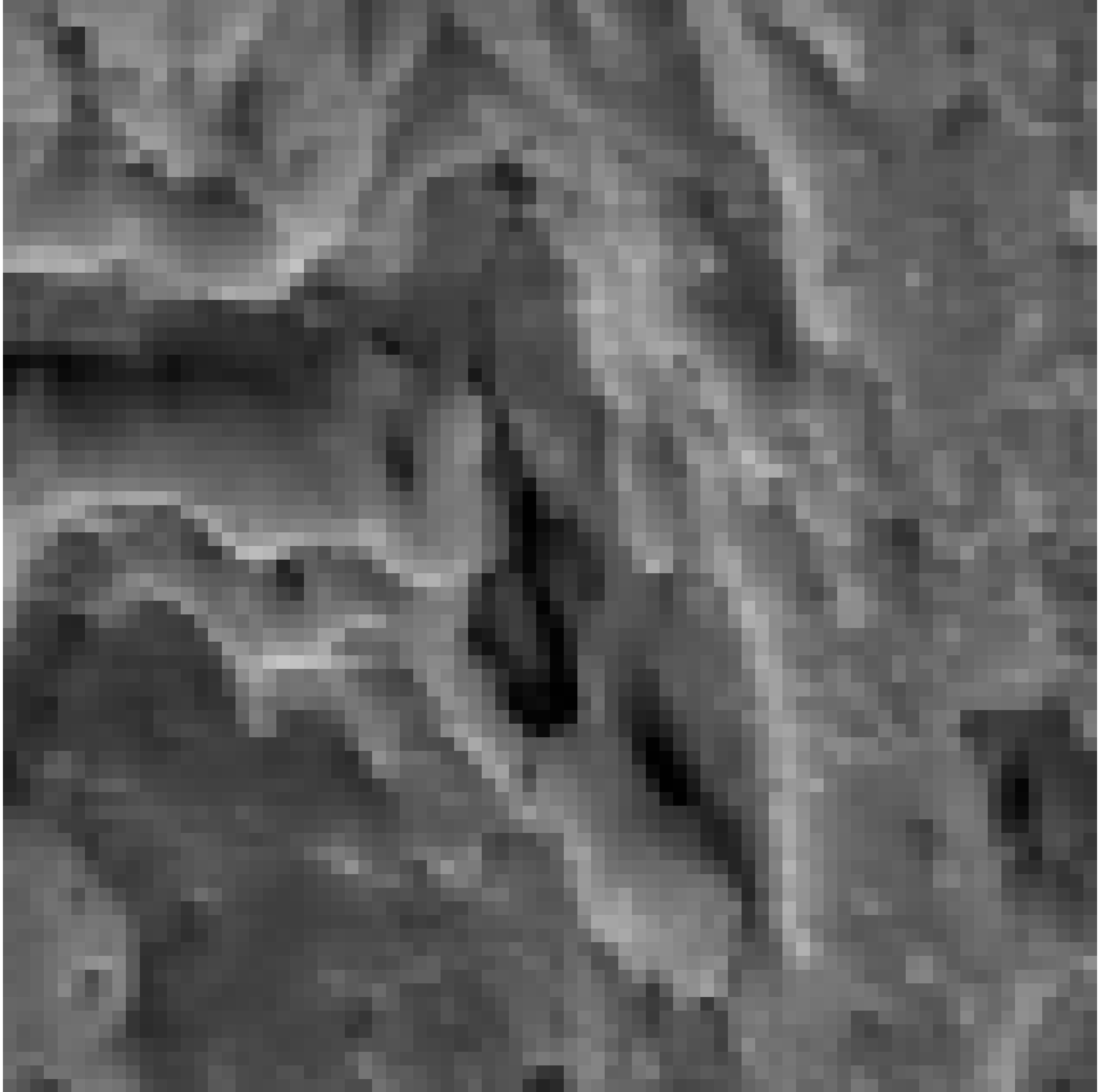

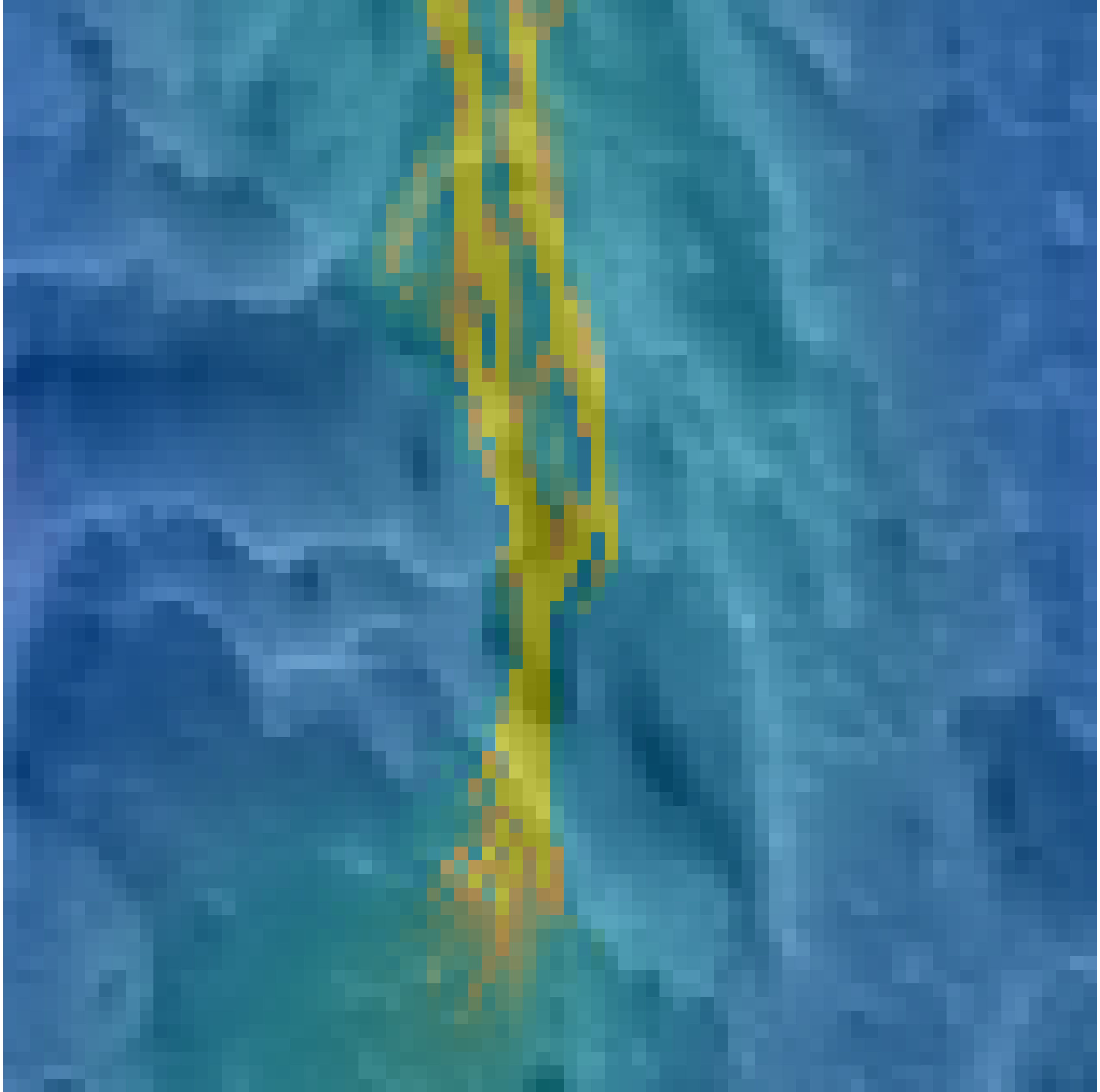

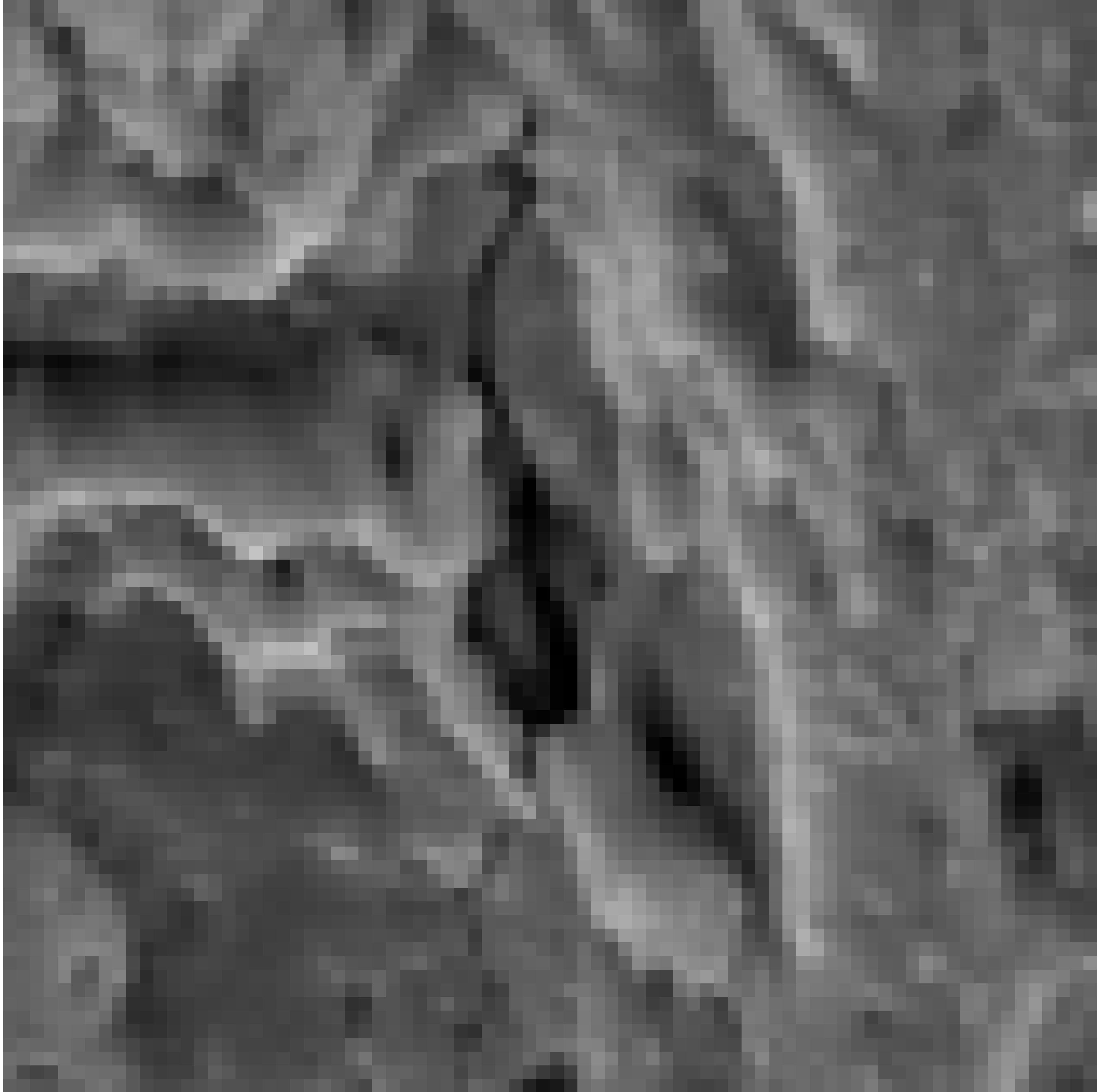

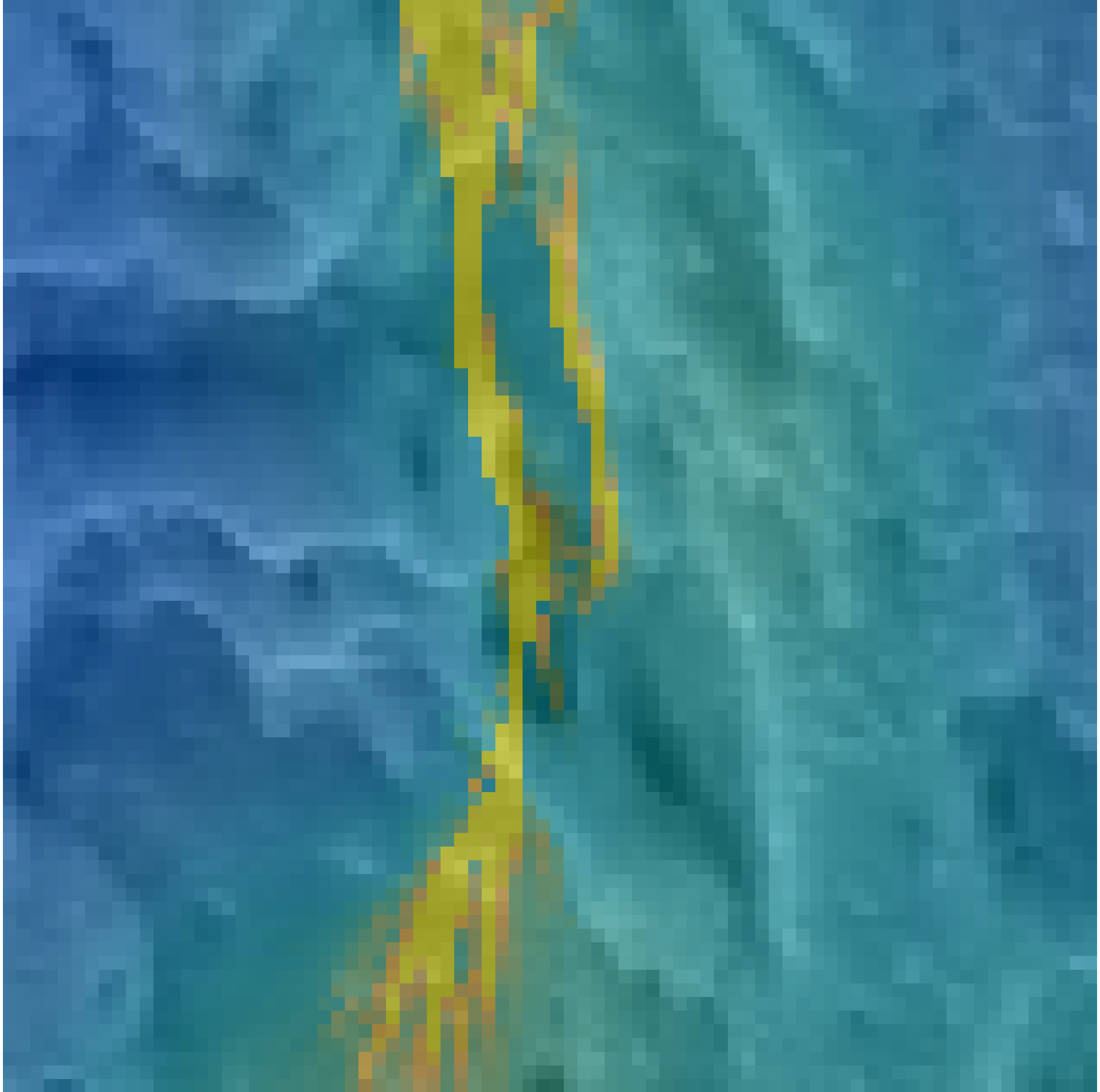

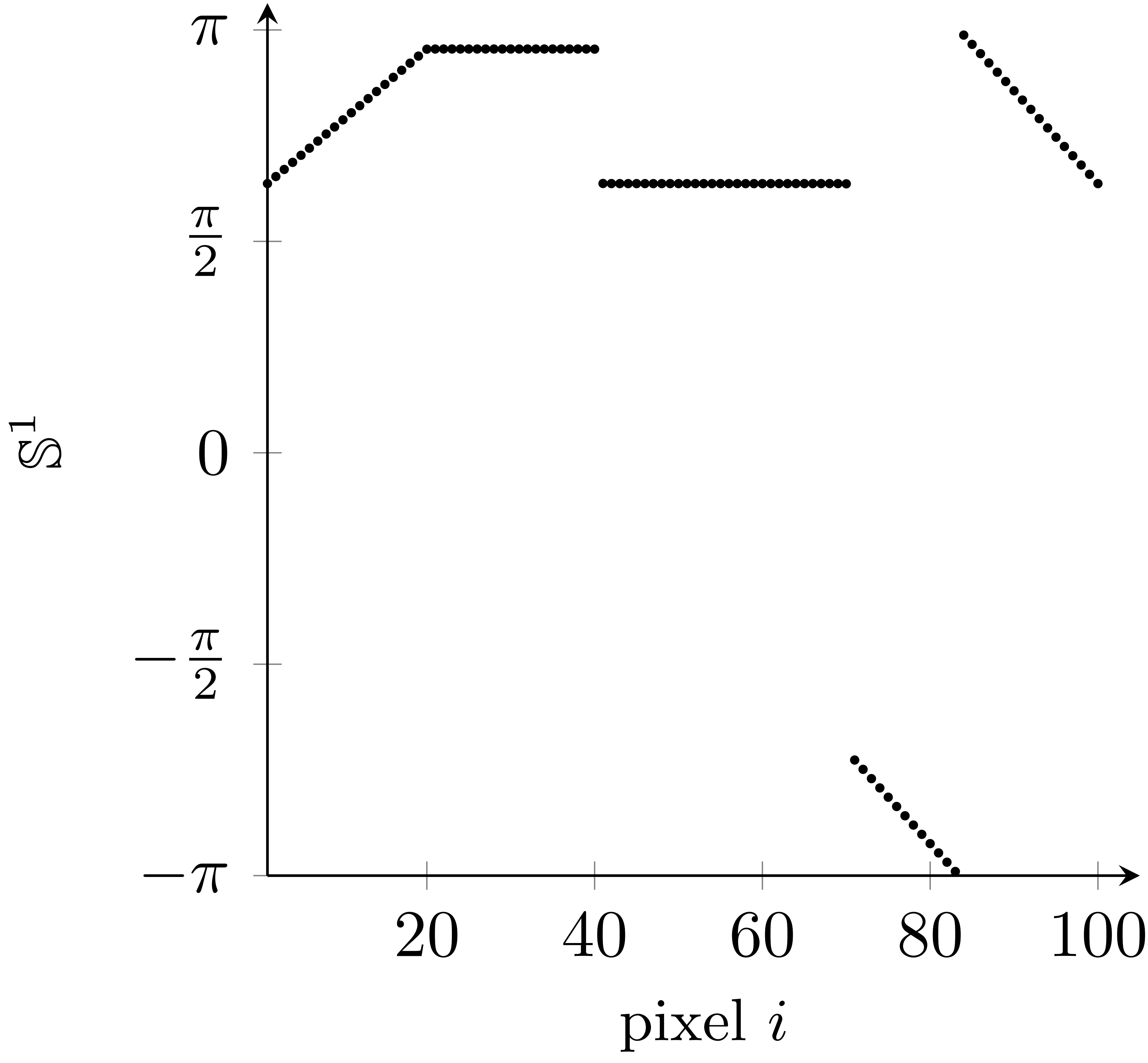

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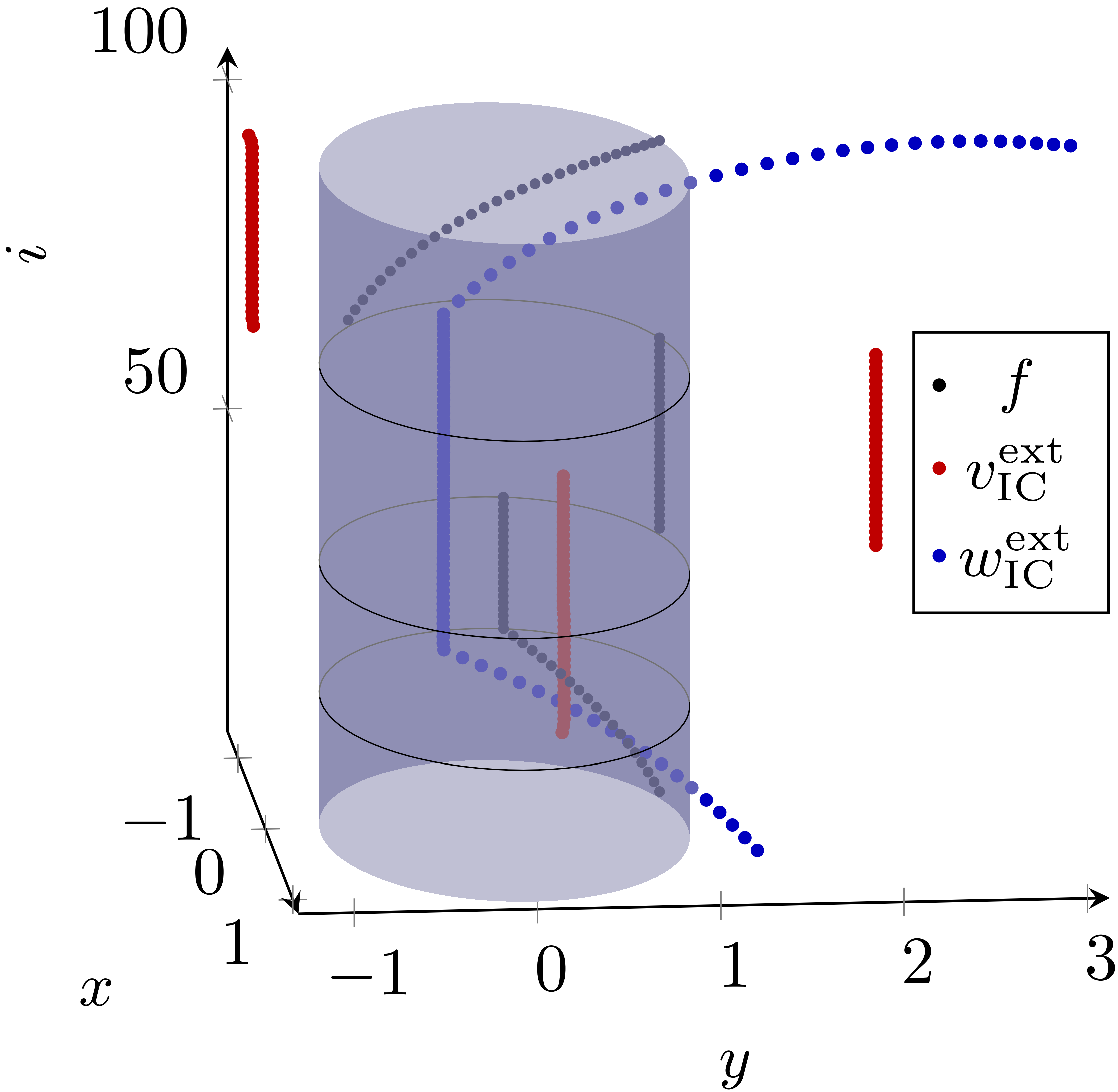

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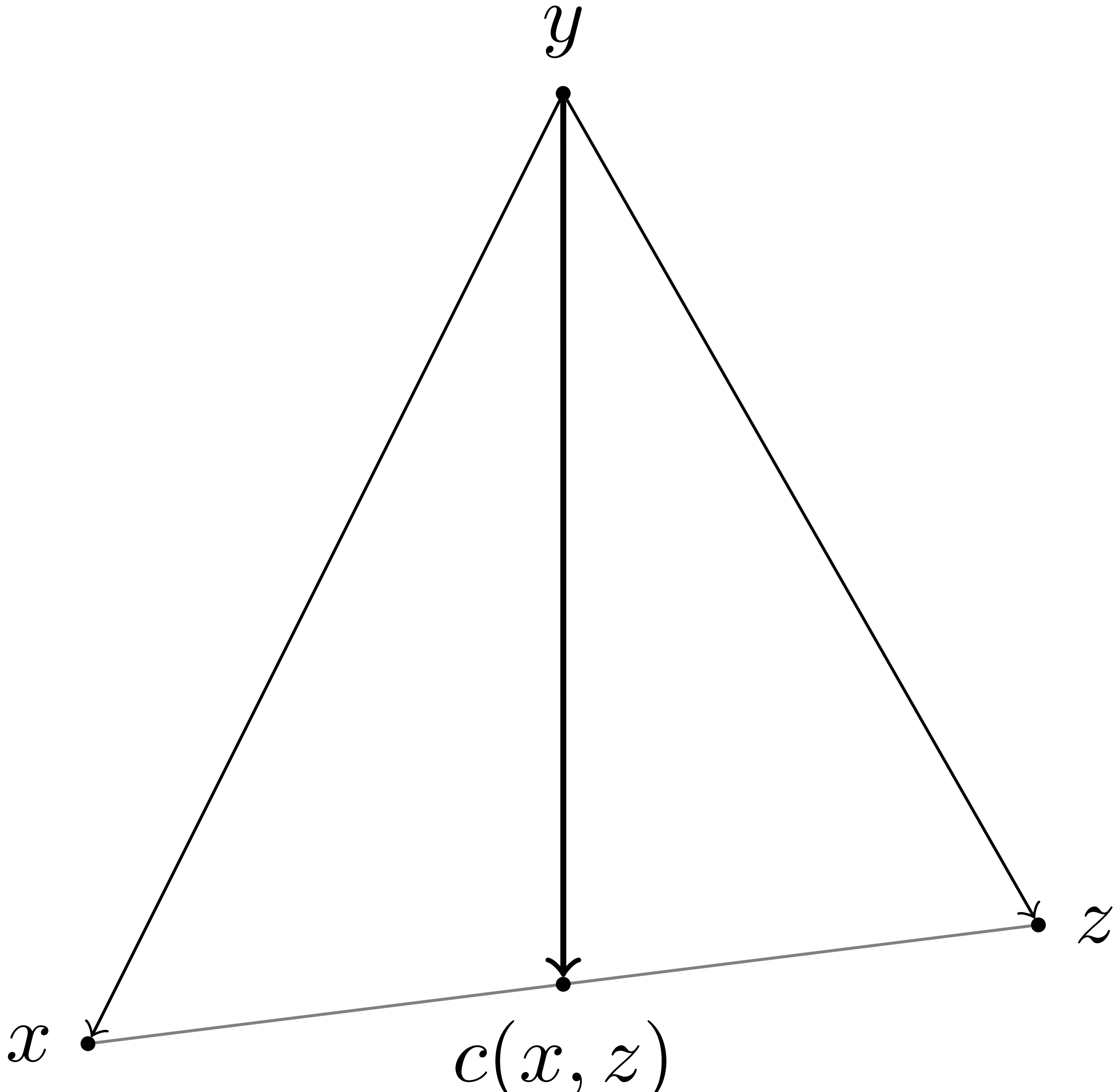
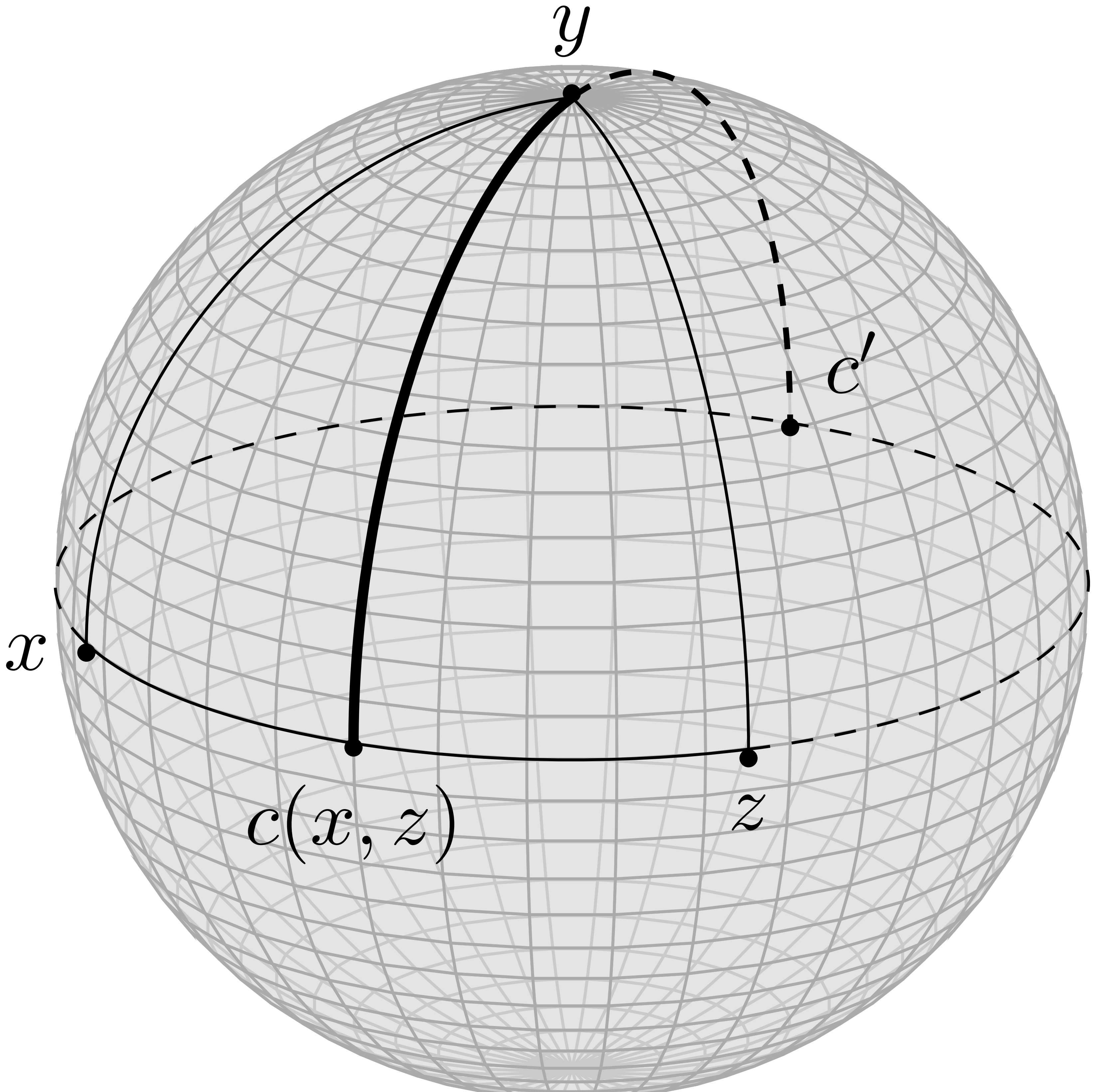
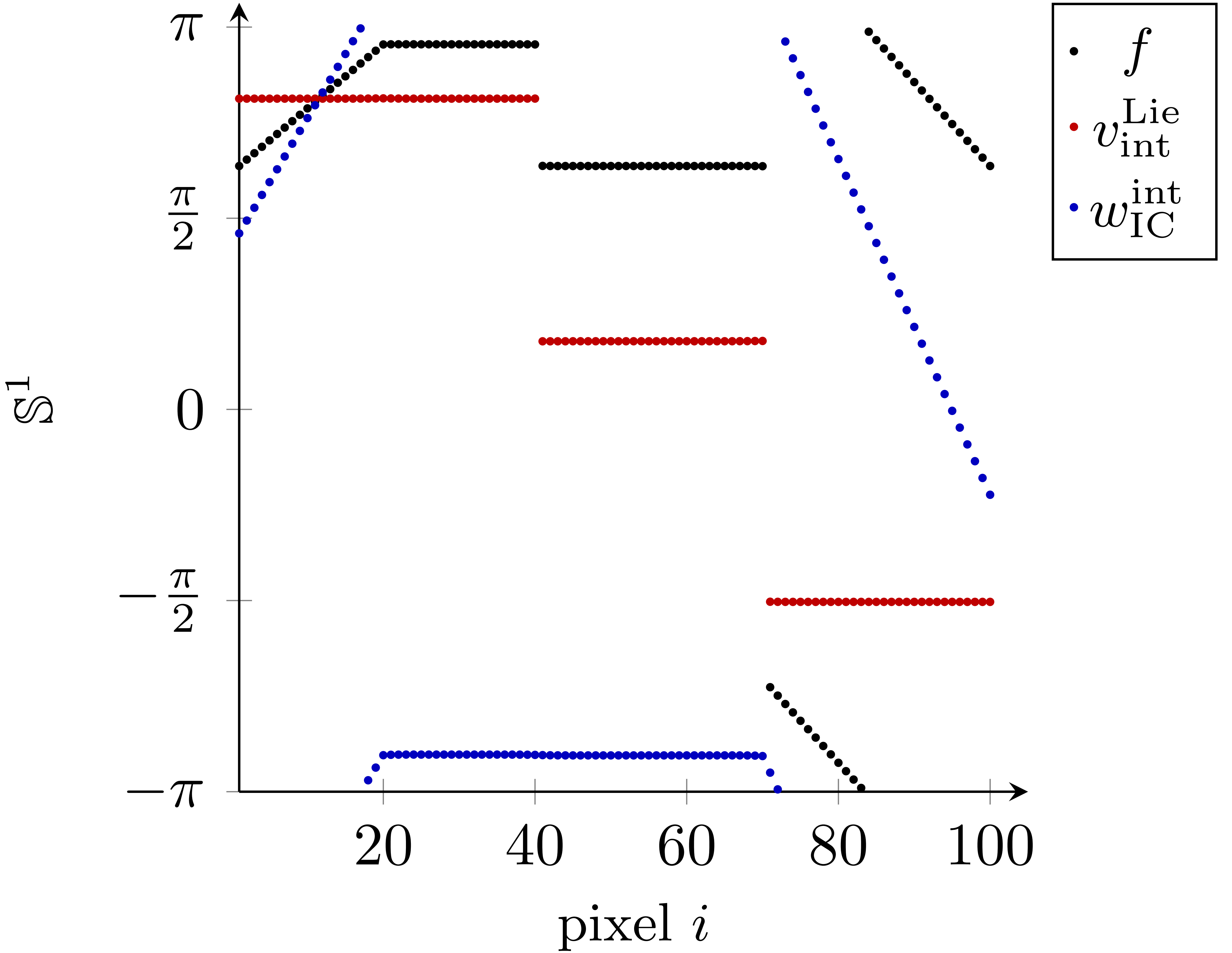
$w_{\mathrm{IC}}^{\mathrm{ext}}$



y
z
x
c(x, z)


y
c′
x
c(x, z)
z



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pixel i
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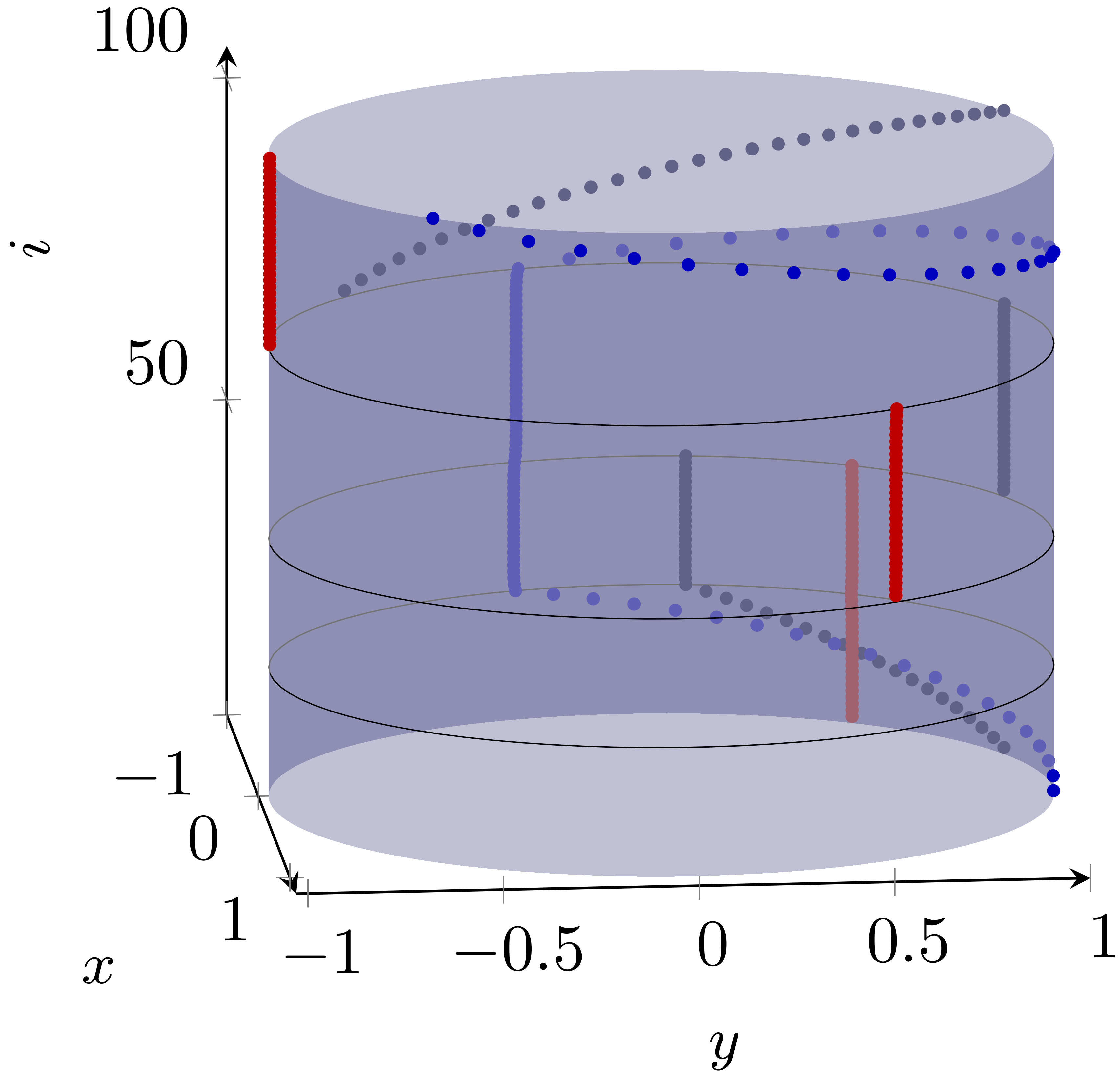
$v_{\text{int}}^{\text{Lie}}$
$w_{\text{IC}}^{\text{int}}$



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0.5
1
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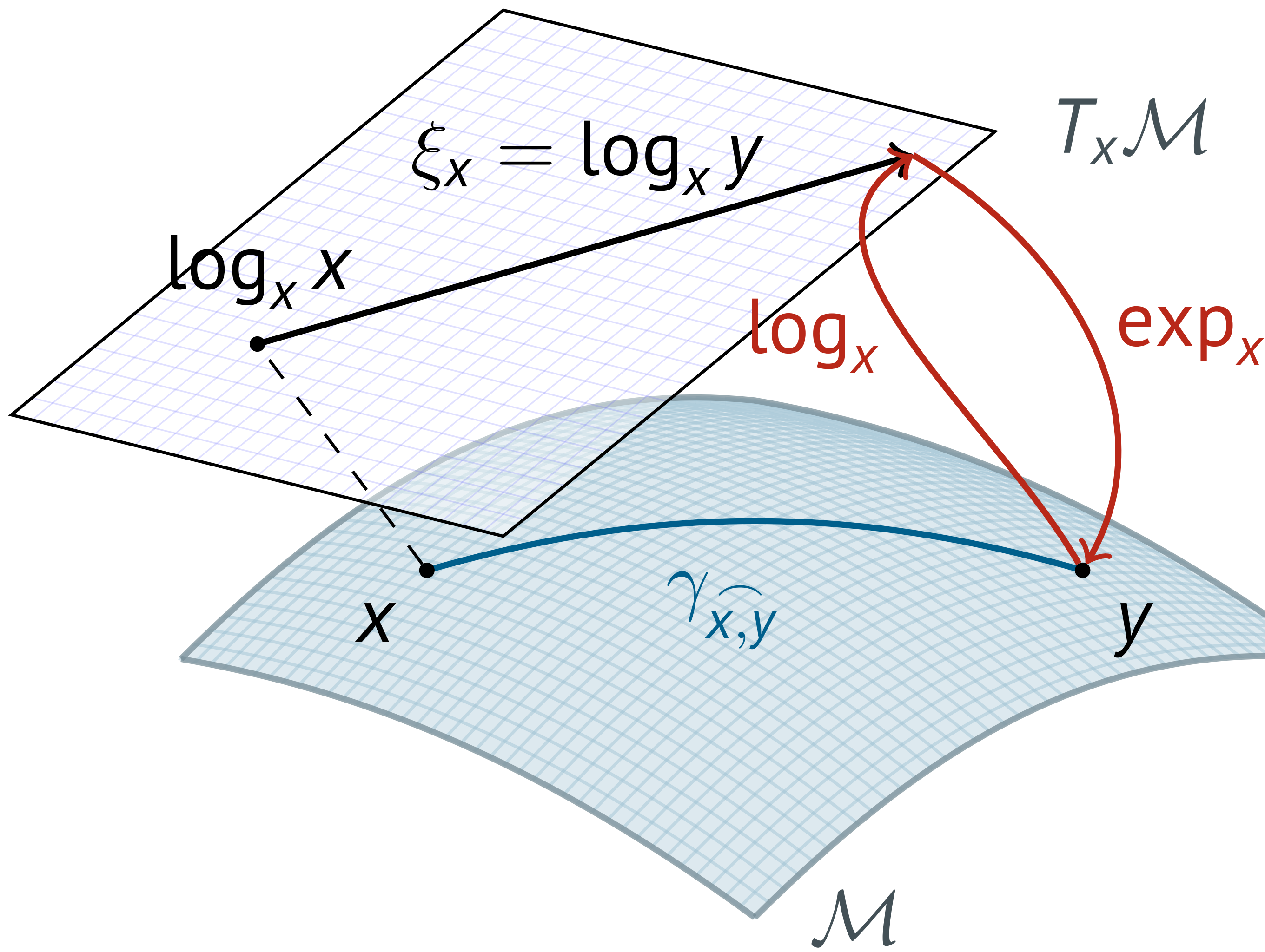
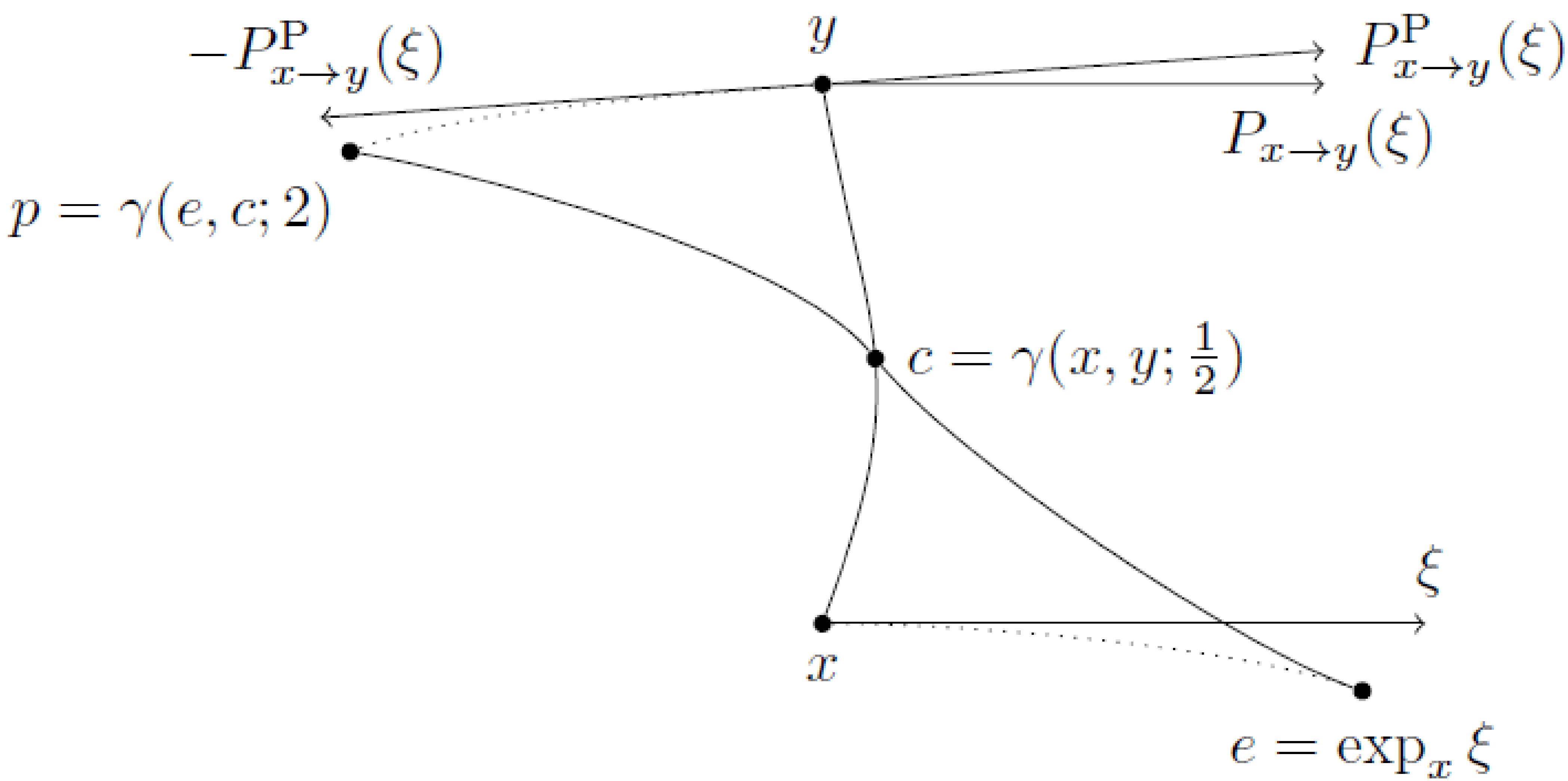
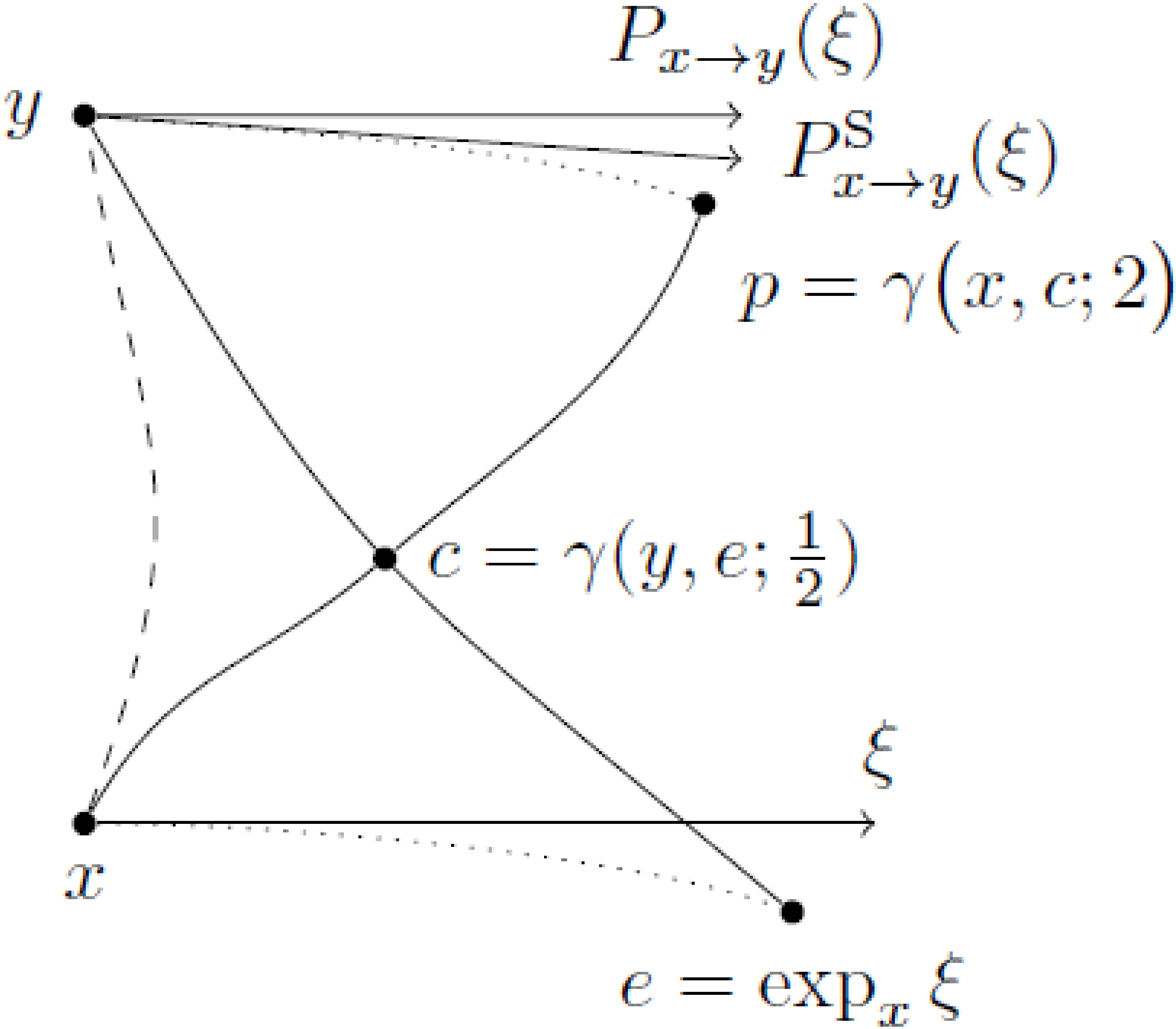


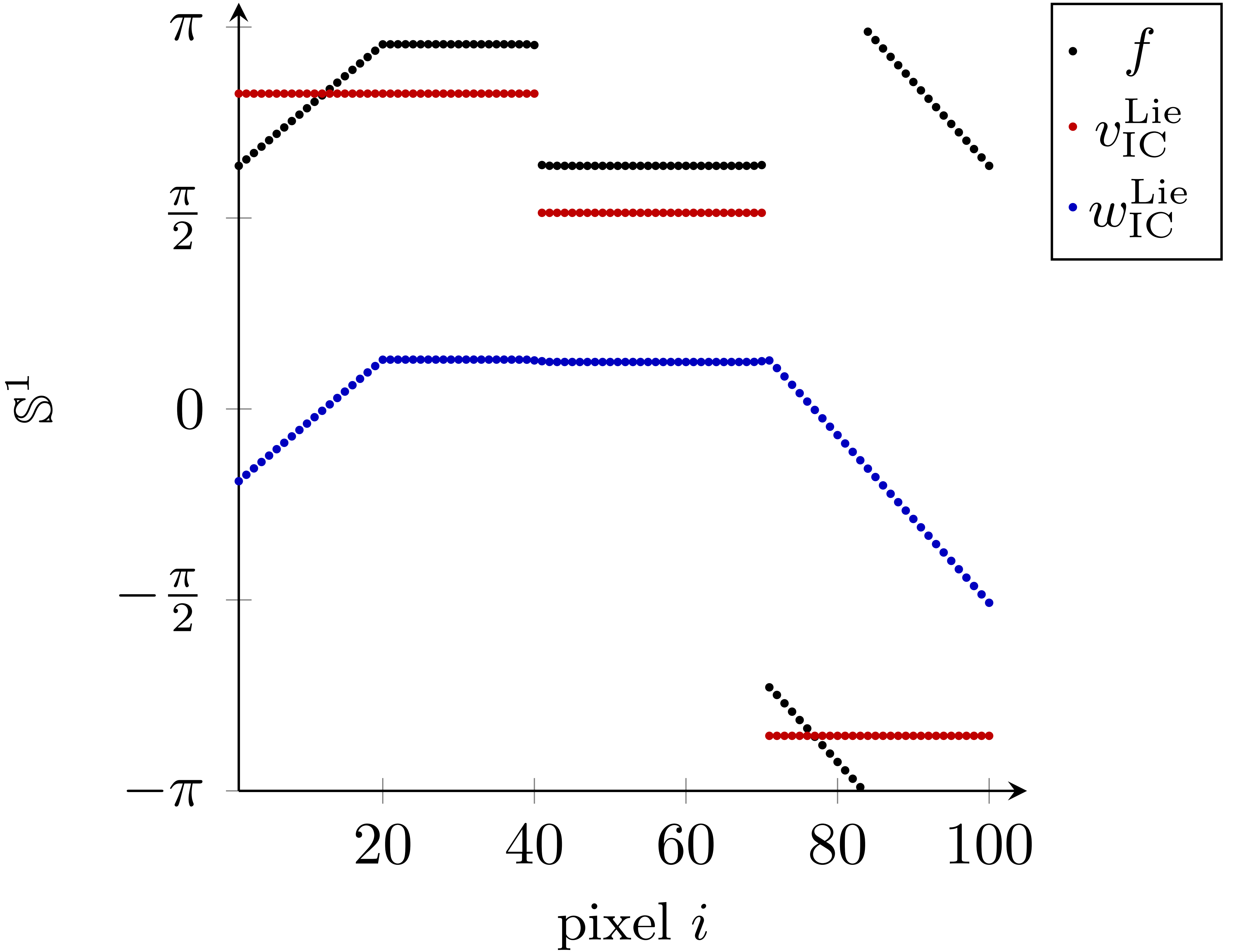
$T_x\mathcal{M}$
$\xi_x = \log_x y$
$\log_x x$
$\log_x$
$\exp_x$
x
$\gamma_{\widehat{x,y}}$
y
$\mathcal{M}$



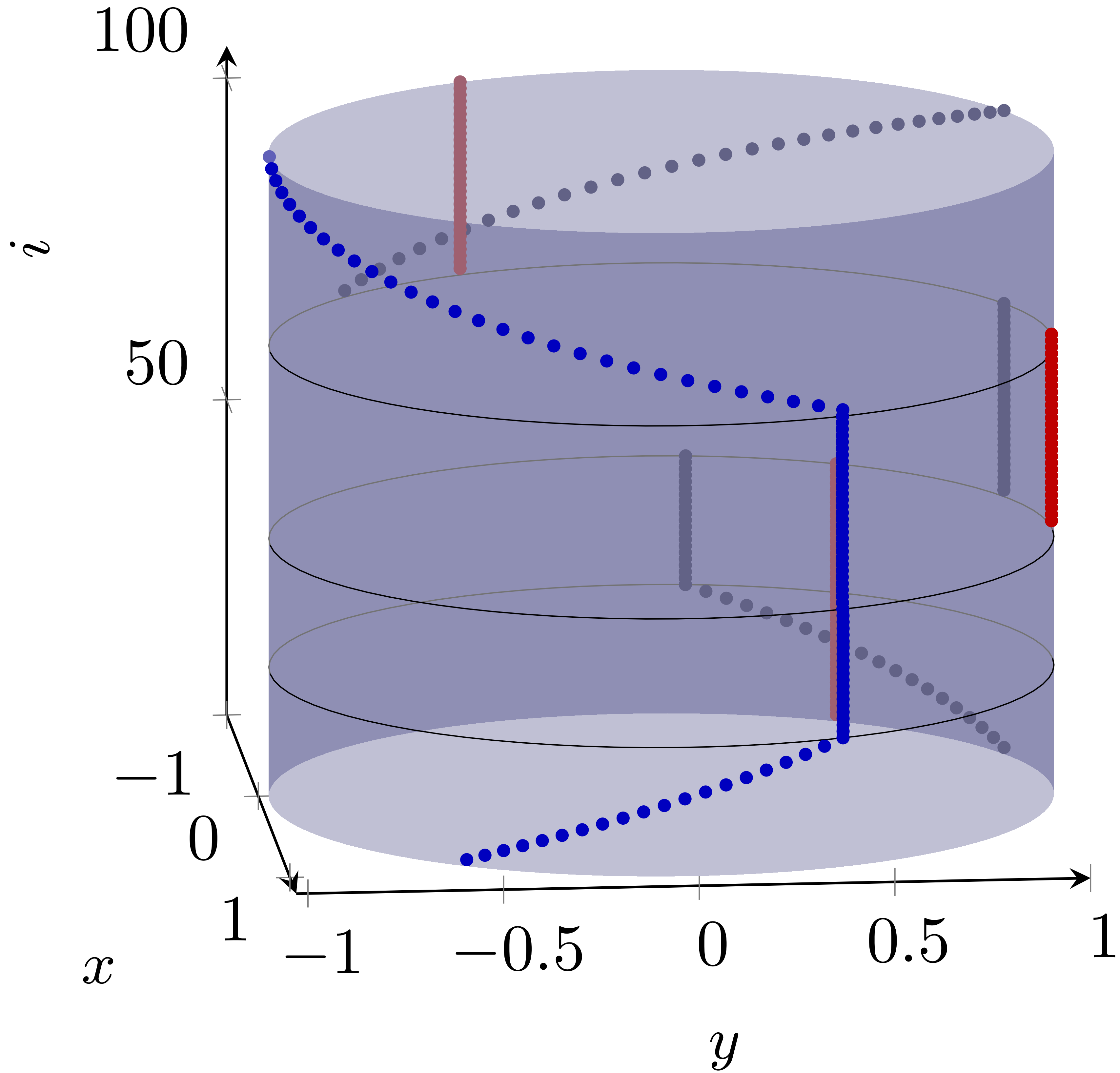
$-P^{\mathrm{P}}_{x\to y}(\xi)$
y
$P^{\mathrm{P}}_{x\to y}(\xi)$
$P_{x\to y}(\xi)$
$p = \gamma(e, c; 2)$
$c = \gamma(x, y; \frac{1}{2})$
ξ
x
$e = \exp_x \xi$

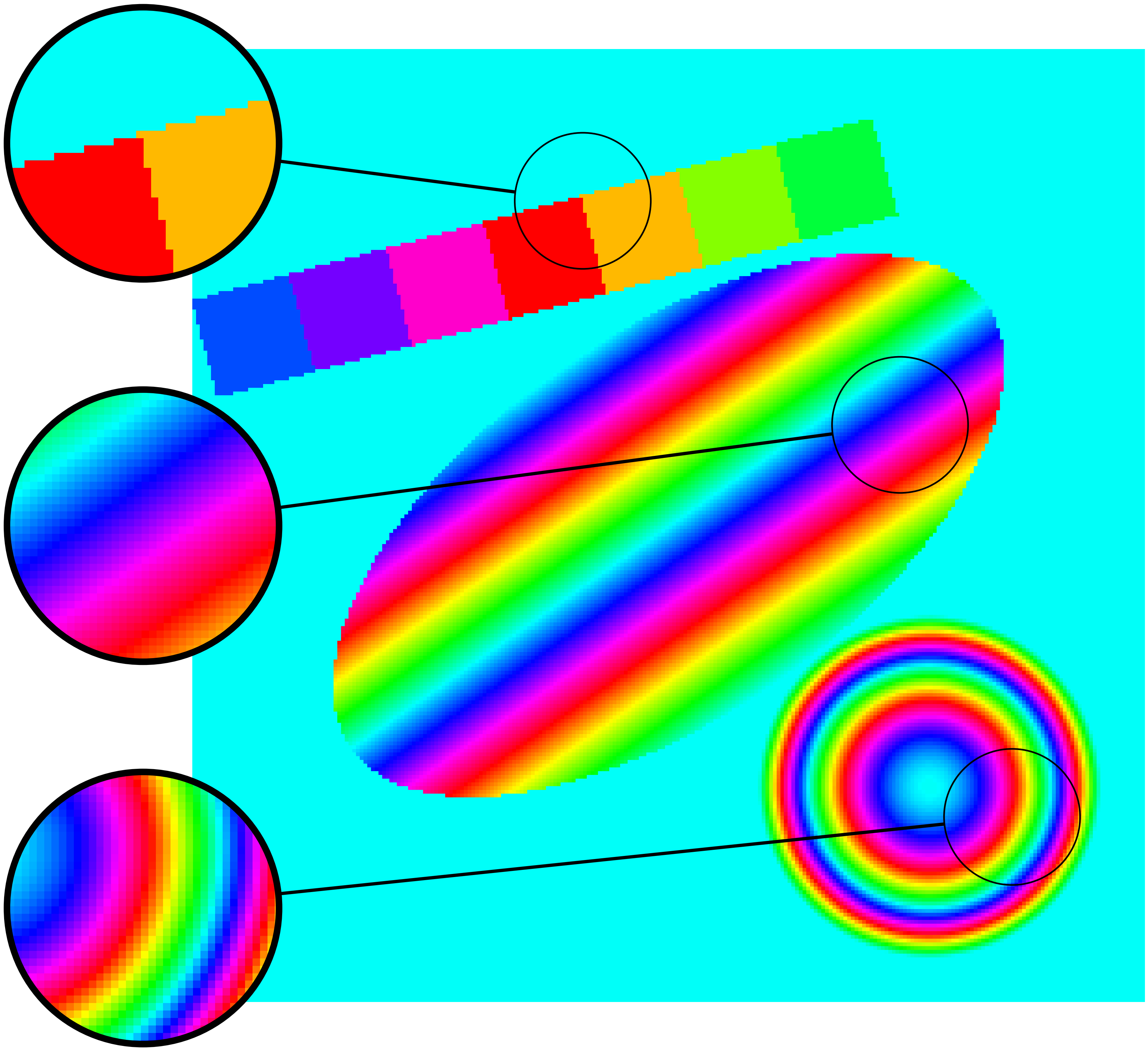


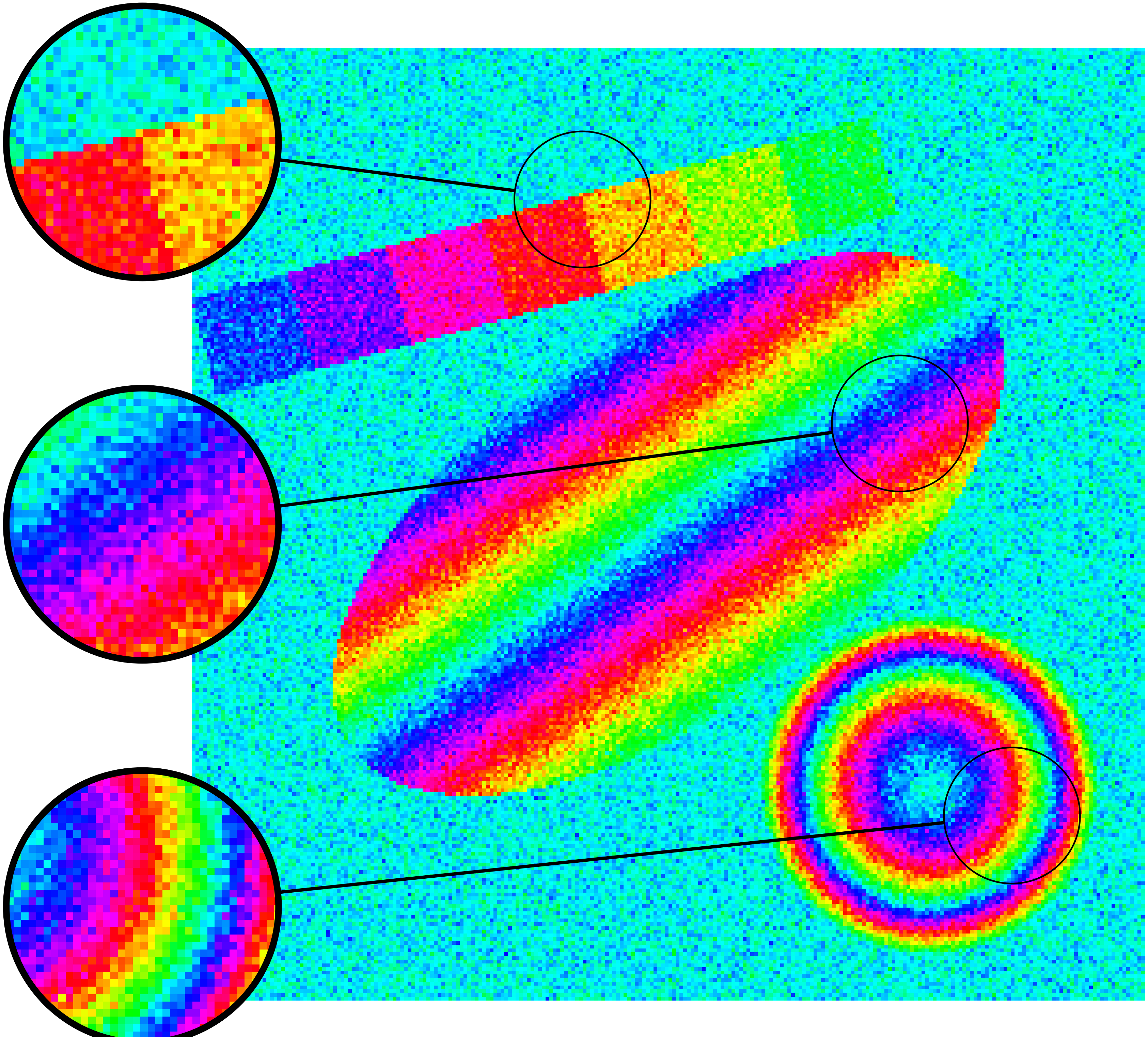
$P_{x\to y}(\xi)$
y
$P^{\mathrm{S}}_{x\to y}(\xi)$
$p = \gamma\big(x, c; 2\big)$
$c = \gamma(y, e; \frac{1}{2})$
ξ
x
$e = \exp_x \xi$

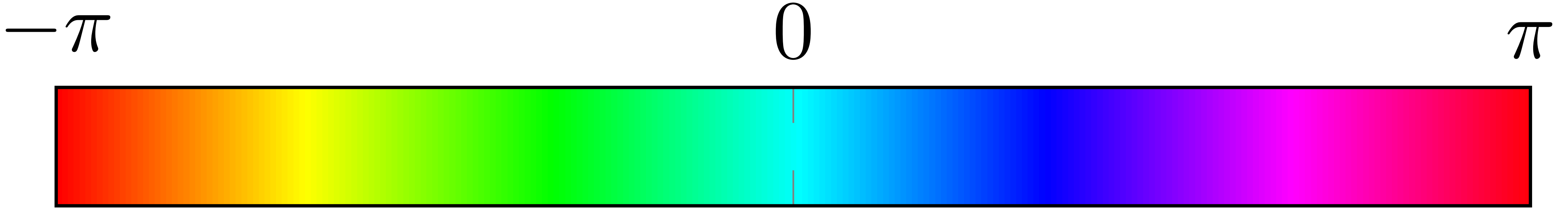


π
$\frac{\pi}{2}$
0
$-\frac{\pi}{2}$
$-\pi$
$\mathbb{S}^1$
20
40
60
80
100
pixel i
f
$v_{\mathrm{IC}}^{\mathrm{Lie}}$
$w_{\mathrm{IC}}^{\mathrm{Lie}}$


100
50
i
−1
0
1
x
−1
−0.5
0
0.5
1
y






$-\pi$
0
π