Surface tension implementation, verification and validation for separated two-phase flows

S. Fortin\textsuperscript{1}, S. Étienne\textsuperscript{1}, C. Béguin\textsuperscript{1}, D. Pelletier\textsuperscript{1}, L. Brosset\textsuperscript{*}

\textsuperscript{1} Mechanical Engineering Department, École Polytechnique de Montréal

\textsuperscript{*} Liquid motion department, GTT (GazTransport&Technigaz)

17 Octobre 2017
1. Surface tension implementation in CADYF

2. Verification and validation
   - Water droplets shapes under the effect of gravity
   - Water droplets oscillation

3. Numerical Simulation of Kelvin-Helmholtz Instability
   - Model presentation
   - Results
Outline

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About CADYF

- Developed at Polytechnique Montréal.
- Stands for *Computer-Assisted DYnamics of Fluids*
- Finite element program for analysis of planar, axi-symmetric, axi-symmetric swirl and three dimensional Navier-Stokes flows.
- Particularities :
  - Transient simulations with adaptive time step and order.
  - Two-phase flows (fluid-structure and fluid-fluid) using interface tracking method.
  - Monolithic resolution between phases and pseudo-solid mesh deformation.
Surface tension implementation in CADYF

Surface tension calculation

- Interface curvature is computed using the following relation:

\[ \kappa = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} \]

where the derivatives are computed using parametric finite differences.

- \( \tau_\sigma \) is applied using reaction method.

\[ \tau_\sigma = \sigma \kappa h_e \]
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Verification and validation
Water droplets shapes under the effect of gravity

Theoretical Background

- Young-Laplace equation.

\[ \gamma(r, r', r'') = \gamma_a + \beta [r_a - \cos(\theta)] \quad \text{with} \quad \beta = \frac{\Delta \rho g}{\sigma} \]

- Solved with ODE45 from Matlab.
Numerical simulation

- Transient simulation with CADYF using the shape from Y-L equation as initial interface shape.
- Displacement of the interface is less than $5.10^{-3}$ (mm) for a mesh of 66 elements at the interface.
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Theoretical Background

- Shape of an oscillating droplet:

\[ r(t, \theta) = \sqrt{R^2 - \frac{R_n(t)^2}{2}} + R_n(t) \cos(n\theta) \]

where \( R \) is the steady droplet radius and \( R_n(t) = r_n \sin(\omega_n t) \) is the oscillation of the droplet.
Theoretical Background (Fundamental Frequency)

- Potential energy of a 2D oscillating droplet:
  \[ E_p = \sigma L_n \rightarrow k_{eq} = \frac{\sigma \pi}{R} (n^2 - 1) \]

- Kinetic energy of a 2D oscillating droplet:
  \[ E_k = \frac{\rho}{2} \int_{0}^{2\pi} \left[ \Phi \frac{d\Phi}{dr} \right]_{r=R} Rd\theta \]
  \[ \rightarrow m_{eq} = \frac{(\rho_1 + \rho_2)\pi R^2}{n} \]

where \( \Phi \) is the velocity potential.

- Frequency of a 2D oscillating droplet:
  \[ \omega_n = \frac{k_{eq}}{m_{eq}} = \frac{\sigma}{(\rho_1 + \rho_2)R^3} (n^3 - n) \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>Fundamental Frequency (Hz)</th>
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<tbody>
<tr>
<td>2</td>
<td>20.8</td>
</tr>
<tr>
<td>4</td>
<td>65.8</td>
</tr>
<tr>
<td>6</td>
<td>123.1</td>
</tr>
<tr>
<td>8</td>
<td>190.7</td>
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</tbody>
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Same relation as Lord Rayleigh (1879).
Verification and validation
Water droplets oscillation

Theoretical Background (Equivalent Damping)

- From the Batchelor formula we find the dissipation power in the global system:

\[ P_d = c_{eq} \dot{\mathcal{R}}_n^2 = 2\mu_1 \int_{\Omega_1} e_{ij}^1 e_{ij}^1 dV + 2\mu_2 \int_{\Omega_2} e_{ij}^2 e_{ij}^2 dV \]

with \[ e_{ij}^k = \frac{1}{2} \left( \frac{\partial u_i^k}{\partial x_j} + \frac{\partial u_j^k}{\partial x_i} \right) \]

which gives us:

\[ \zeta = (2n - 2) \sqrt{\frac{n}{n^2 - 1}} \text{Oh} \]

avec \[ \text{Oh} = \frac{2\pi (\mu_1 + \mu_2)}{(\rho_1 + \rho_2) R \sigma} \]

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Numerical Simulation

- Transient simulation with CADYF (water/air) using the fully deformed interface as initial shape.
- 23 000 stabilized P1-P1 elements (80 on the interface) was found satisfactory.
Numerical Simulation : Results

Verification and validation
Water droplets oscillation

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MULTIPHASE 2017
Polytechnique Montréal 11 / 23
Numerical Simulation : Results

Verification and validation
Water droplets oscillation
Numerical Simulation : Animation

9 times slowed down
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Thorpe’s experiment

- Comparison with Thorpe’s experiment (1969):
  - $\rho_1 = 780 \text{kg/m}^3$
  - $\mu_1 = 1.5 \text{mPa/s}$
  - $\rho_2 = 1000 \text{kg/m}^3$
  - $\mu_2 = 1.0 \text{mPa/s}$
  - $\sigma = 0.04 \text{N/m}$
  - $\theta = 4.1^\circ$
Numerical Simulation of Kelvin-Helmholtz Instability

Model presentation

Dimensional analysis

- 5 dimensionless numbers \((U_r = \sqrt{gH})\):

\[
\frac{\rho_1}{\rho_2}, \frac{\mu_1}{\mu_2}, \theta; \quad \frac{\rho_1 U_r^2 H}{\sigma} = We; \quad \frac{\rho_1 U_r H}{\mu_1} = Re
\]

- Dimensionless amplitude:

\[
\alpha^* = \frac{\alpha}{H} \text{ with } \alpha = \max[y(x_i)] \text{ where } x_i \in [L/3, 2L/3]
\]

- Dimensionless time:

\[
t^* = t \sqrt{\frac{g}{H}}
\]

- Thorpe’s experiment:

\[
We = 689 \text{ and } Re = 23937
\]
Remeshing
Surface Tension
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Results comparison

![Graph showing comparison of amplitude growth over solution time for different methods: linear theory, experimental, CADYF (~166,000 elements), and Strubelj (219,500 elements).]
Results comparison

- Most unstable wave length:
  - Linear Theory $\lambda_c = 27 \text{mm}$
  - Experimental $\lambda_c = 25 - 45 \text{mm}$
  - Numerical $\lambda_c \approx 38 \text{mm}$
  - Štrubelj $\lambda_c \approx 40 \text{mm}$

- Instability seems to appear at a relative velocity of 0.28 m/s which is larger than linear theory prediction ($U_c \approx 0.2 \text{m/s}$)
Vorticity field

- $t = 2.1$ s
- $t = 2.2$ s
- $t = 2.3$ s
- $t = 2.4$ s
- $t = 2.5$ s
- $t = 2.6$ s
Numerical Simulation of Kelvin-Helmholtz Instability

Results

Similarity

\[ \alpha^* = 0.075 \]
\[ t^* = 28.6 \]
\[ \lambda^* = 0.59 \]
\[ \text{We} = 689, \text{Re} = 31\,504 \]

\[ \alpha^* = 0.63 \]
\[ t^* = 29.4 \]
\[ \lambda^* = 0.63 \]
\[ \text{We} = 689, \text{Re} = 23\,937 \]

\[ \alpha^* = 0.075 \]
\[ t^* = 34.0 \]
\[ \lambda^* = 0.91 \]
\[ \text{We} = 203, \text{Re} = 23\,937 \]

\[ \alpha^* = 0.59 \]
\[ t^* = 29.7 \]
\[ \lambda^* = 0.59 \]
\[ \text{We} = 689, \text{Re} = 23\,937, H/2 \]
References

Questions?