DE LA RECHERCHE À L'INDUSTRIE

What is the final size of turbulent mixing zones driven by the Faraday instability?

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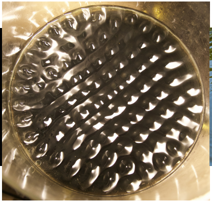
The Faraday instability

Appearing at periodically accelerated interfaces, discovered by Faraday (1831)

$$G(t) = G_0(1 + F \sin \omega t)$$


Linear analysis based on Floquet decomposition see Benjamin and Ursell (1954), Kumar and Tuckerman (1994).

A sub-harmonic response is commonly observed, corresponding to $\omega/2$ oscillations of the interface.

Studying the turbulent Faraday problem

The Faraday problem is mostly studied in the context of laminar flows.

However, for miscible fluids, if the forcing F is strong enough or for sufficiently disordered initial conditions, standing waves interact each other leading to a turbulent mixing zone (MZ).

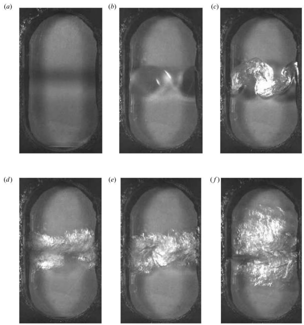


FIGURE 5. Image sequence of interface evolution during an experiment; (a) $t = 0$ s, (b) $t = 4.98$ s, (c) $t = 6.32$ s, (d) $t = 6.74$ s, (e) $t = 6.98$ s and (f) $t = 11.68$ s. Experimental parameters: $A = 10$ cm, $f = 1.5$ Hz and $t_0 = 75$ min.

Miscible Faraday

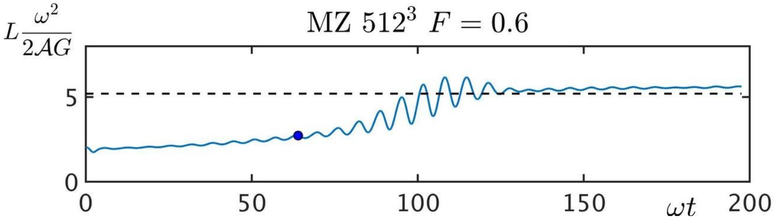
Zoueshtiagh et al. JFM (2009)

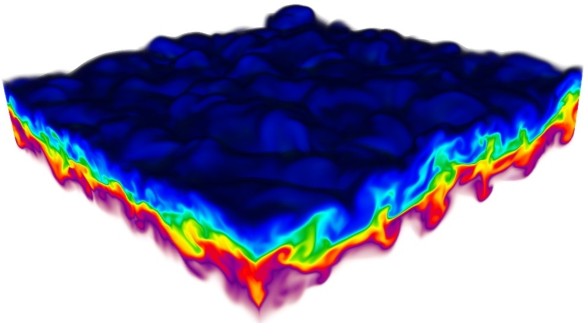
see also Diwakar et al. PoF (2015)

for instability onset

As the MZ grows, resonance conditions are no longer fulfilled. This leads to a saturation of the MZ 's width together with a decay of turbulence intensity.

What is the final size ?





Dimensional analysis

Neglecting the influence of initial conditions and dissipative phenomena, the only control parameters in this problem are ω, $\mathcal{A}$, G_0, F leading to :

$$L_{\mathrm{sat}} = \frac{2\mathcal{A}G_0}{\omega^2}\mathcal{G}(F)$$

- (I) Theoretical framework

  Mixing zone (MZ), stratified homogeneous turbulence (SHT) and rapid acceleration (RA) model

- (II) Gravity waves and mean density profile

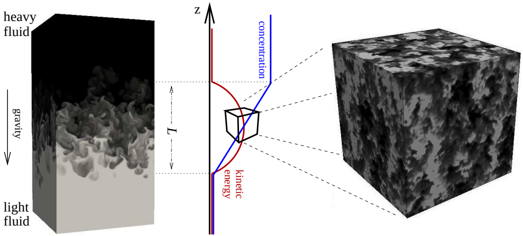
  Weakly non linear analysis and derivation of a saturation criterion

- (III) Verification of the saturation criterion

  Simulations of homogeneous (SHT) and inhomogeneous (MZ) stratified flows

- (I) Theoretical framework

  Mixing zone (MZ), stratified homogeneous turbulence (SHT) and rapid acceleration (RA) model



Stratified Homogeneous Turbulence (SHT)

Mixing Zone (MZ)

$$\partial_t C + (\mathbf{U}\cdot\nabla)C = \mathcal{D}\Delta C,$$
$$\partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\nabla P + 2\mathcal{A}CG(t)\mathbf{n}_3 + \mu\Delta\mathbf{U},$$
$$\nabla\cdot\mathbf{U} = 0,$$

SHT:

$$\partial_t c + (\mathbf{u}\cdot\nabla)c = \frac{1}{L}u_3 + \mathcal{D}\Delta c,$$
$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + 2\mathcal{A}cG(t)\mathbf{n}_3 + \mu\Delta\mathbf{u},$$
$$\nabla\cdot\mathbf{u} = 0,$$
$$\dot{L} = \frac{12}{L}\ell_0\left\langle u_3 c\right\rangle,$$

See Griffond et al. JFE (2014), Soulard et al. PoF (2014), Burlot et al. JFM (2015)

(I) The rapid acceleration (RA) model

A simple model for vertical velocity $\mathcal{U}(t,\theta)$ and buoyancy coefficient $\mathcal{B}(t,\theta)$ cf Gréa. PoF, (2013).

$$\begin{aligned}
\partial_t \mathcal{U} &= -2\mathcal{A}G(t)\sin^2(\theta)\mathcal{B}, \\
\partial_t \mathcal{B} &= \frac{1}{L}\mathcal{U}, \\
\dot{L} &= \frac{12}{L}\int_0^{\pi} \mathcal{U}\mathcal{B}\sin(\theta)d\theta
\end{aligned}$$

In short, this is RDT, Hanazaki & Hunt, JFM (1996) + equation for L

It is convenient to write it on the form :

$$\begin{aligned}
&\partial_{tt}\mathcal{B}(t,\theta) + \frac{\dot{L}(t)}{L(t)}\partial_t\mathcal{B}(t,\theta) + \frac{2\mathcal{A}G(t)\sin^2(\theta)}{L(t)}\mathcal{B}(t,\theta) = 0, \\
&L(t) = 6\int_0^{\pi} \mathcal{B}^2(t,\theta)\sin(\theta)d\theta + L_0.
\end{aligned}$$

- (II) Gravity waves and mean density profile

  Weakly non linear analysis and derivation of a saturation criterion

Considering $B(t', \theta) = \mathcal{B}(t, \theta)/(6L_0)^{1/2}$ and $t' = N_0 t$ with $N_0 = (2\mathcal{A}G_0/L_0)^{1/2}$ we get :

$$\partial_{tt} B(t,\theta) - 2\frac{\dot{\Omega}(t)}{\Omega(t)}\partial_t B(t,\theta) + \sin^2(\theta)\Omega^2(t)\left(1 + F\cos(\omega/N_0 t)\right)B(t,\theta) = 0,$$

$$\Omega^2(t) = \frac{1}{1 + \int_0^\pi B^2(t,\theta)\sin(\theta)d\theta}.$$

Non linear system of second order equations.

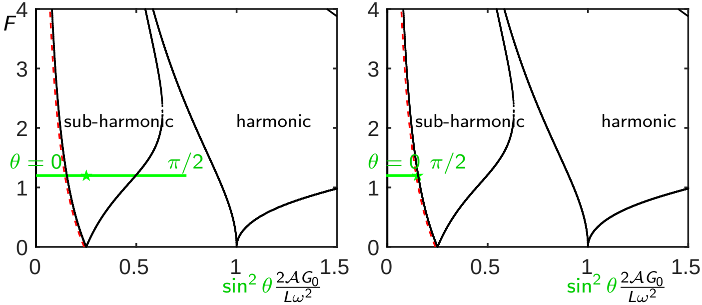
This relatively simple model describes how gravity waves interact with the mean density profile.

Linearising, we obtain for $\theta \in [0\ \pi]$:

$$\partial_{tt} B(t,\theta) + \sin^2(\theta)\left(1 + F\cos(\omega/N_0 t)\right)B(t,\theta) = 0.$$

A Mathieu equation for each angle θ which stability is expressed in the phase diagram.
Studied by Benielli and Sommeria JFM 1998 (Merci Benjamin !)

(II) Mechanism in the stability diagram



Frequencies of gravity waves

$$\sin(\theta)\left(\frac{2\mathcal{A}G_0}{L}\right)^{1/2}$$

Idea : Saturation if all the gravity waves are stable ?

(II) Multiple-scale analysis

$$\partial_{tt}B(t,\theta) - 2\frac{\dot{\Omega}(t)}{\Omega(t)}\partial_t B(t,\theta) + \sin^2(\theta)\Omega^2(t)\left(1 + F\cos(\omega/N_0 t)\right)B(t,\theta) = 0,$$

$$\Omega^2(t) = \frac{1}{1 + \int_0^\pi B^2(t,\theta)\sin(\theta)d\theta}.$$

Same expansion for B :

$$B(t,\theta) = \varepsilon\left(B^{(0)}(t,\tau,\theta) + \varepsilon B^{(1)}(t,\tau,\theta) + \varepsilon^2 B^{(2)}(t,\tau,\theta) + \varepsilon^3 B^{(3)}(t,\tau,\theta) + ...\right),$$

with $\tau = \varepsilon^2 t$ (we use steepest descent method to approximate integrals).

We assume further see Godreche & Manneville (2005)

$$N_0^2/\omega^2 - 1/4 = \varepsilon^2\Delta \quad \& \quad F = \varepsilon^2 f$$

Close to first resonance condition with small forcing

(II) Multiple-scale analysis

At leading order

$$\partial_{tt} B^{(0)}(t,\tau,\theta) + \sin^2(\theta) B^{(0)}(t,\tau,\theta) = 0, \text{ at order } \varepsilon,$$
$$\partial_{tt} B^{(1)}(t,\tau,\theta) + \sin^2(\theta) B^{(1)}(t,\tau,\theta) = 0, \text{ at order } \varepsilon^2.$$

General solution on the form :

$$B^{(0)}(t,\tau,\theta) = a(\tau,\theta) e^{i\sin(\theta)t} + a^*(\tau,\theta) e^{-i\sin(\theta)t},$$
$$B^{(1)}(t,\tau,\theta) = b(\tau,\theta) e^{i\sin(\theta)t} + b^*(\tau,\theta) e^{-i\sin(\theta)t}.$$

At order ε^3

$$\partial_{tt} B^{(2)} + \sin^2(\theta) B^{(2)} = -2\partial_{t\tau} B^{(0)} + 2\sin^2\theta B^{(0)} \Lambda(\tau) - f\sin^2\theta B^{(0)} \sin(\omega/N_0 t)$$

focusing on $\theta = \pi/2$:

$$-2i\partial_\tau a(\tau) + 2a(\tau)\Lambda(\tau) - \frac{1}{2} f a^*(\tau) e^{-4i\Delta\tau} = 0$$

...with stationary solutions corresponding to

$$\frac{L_{\rm sat}}{L_0} = \frac{N_0^2}{\omega^2}(2F+4)$$

We have just demonstrated that for $F \ll 1$ and close to the first resonance condition that :

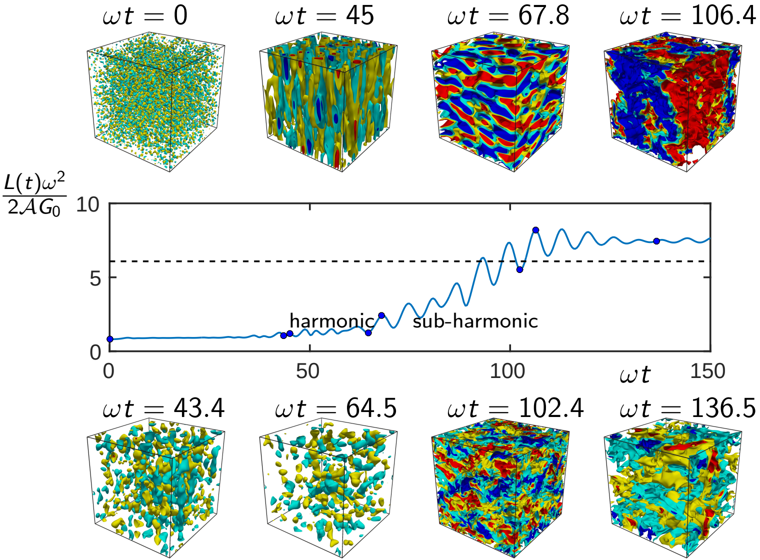
$$L_{\mathrm{sat}} = \frac{2\mathcal{A}G_0}{\omega^2}(2F + 4) \approx \frac{2\mathcal{A}G_0}{\omega^2}\mathcal{G}(F)$$

where $\mathcal{G}$ corresponds to the first transition curve in the Mathieu stability diagram.
Is this result more general ?

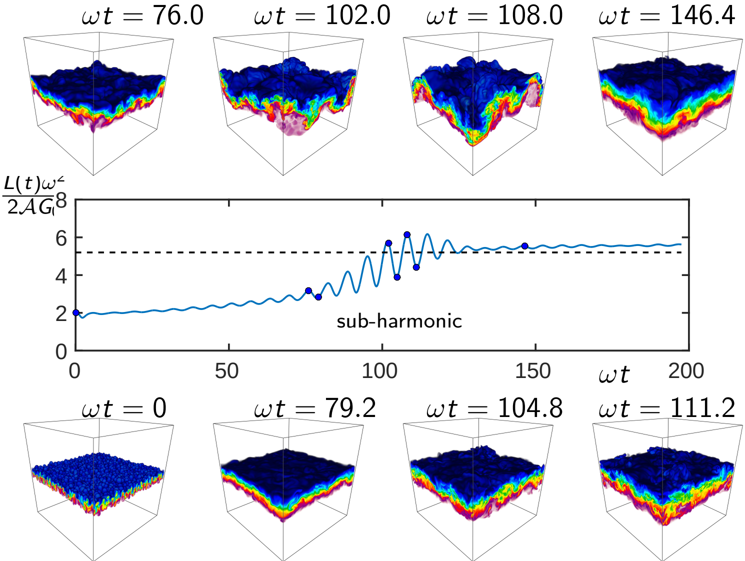
- (III) Verification of the saturation criterion

  Simulations of homogeneous (SHT) and inhomogeneous (MZ) stratified flows

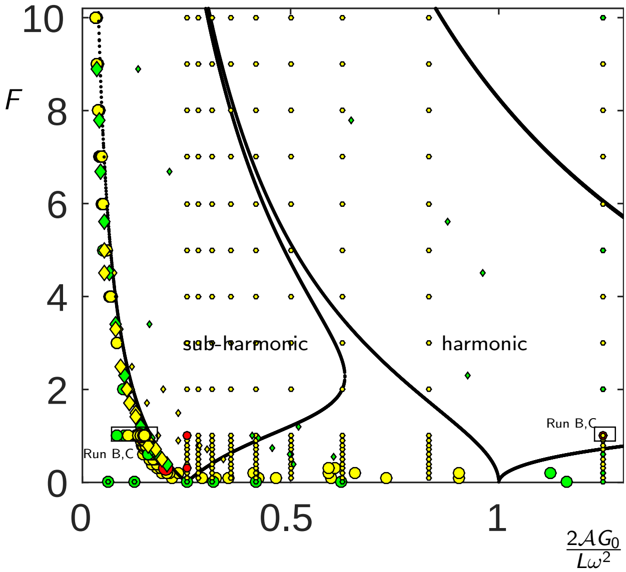
(III) SHT 512^3 simulation with $F = 1$



(III) MZ 512^3 simulation with $F = 0.6$



(III) Phase diagram for SHT and MZ simulations



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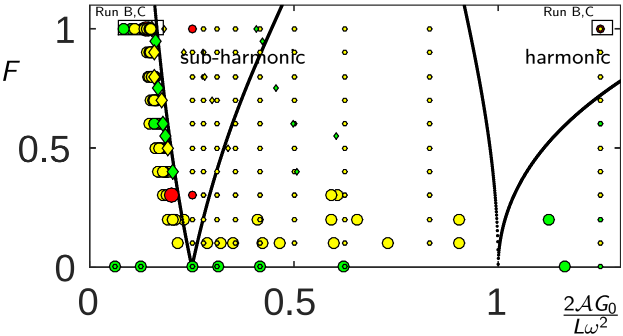

Relatively weak influence of the initial Froude number.
Instability triggered for $F > 0.2 - 0.3$ due to eddy viscosity ? (see also Falcon et al. PRE 2009)

Summary

- An weakly non-linear analysis of the mechanism describing interactions between gravity waves and mean density profile is performed.
- We derive a saturation criterion for final states of MZ driven by the Faraday instability.
- The prediction for Ls is verified against more than 300 MZ and SHT simulations.
- We identify harmonic/ sub-harmonic transition.
- Experimental verifications ?

Paper submitted to JFM

Thank you