Modelling and Numerical Simulations of Contacts in Particle-Laden Flow

Baptiste Lambert[1]

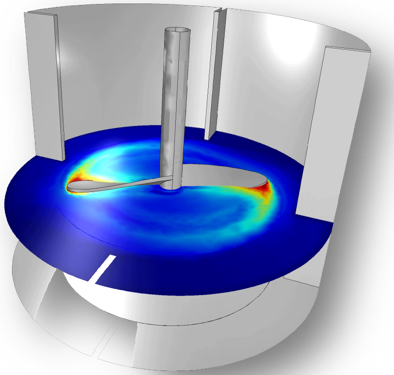
Michel Bergmann
Lisl Weynans

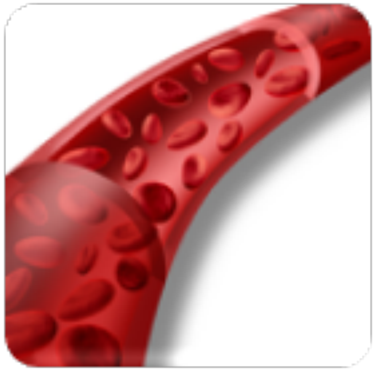
MULTIPHASE

18/10/17
Cachan

[1] Institut de mathématique de Bordeaux & INRIA
baptiste.lambert@inria.fr

Particle Laden Flow: Aims and Applications





I. Definitions and Framework

II. The Local Lubrication Correction Model

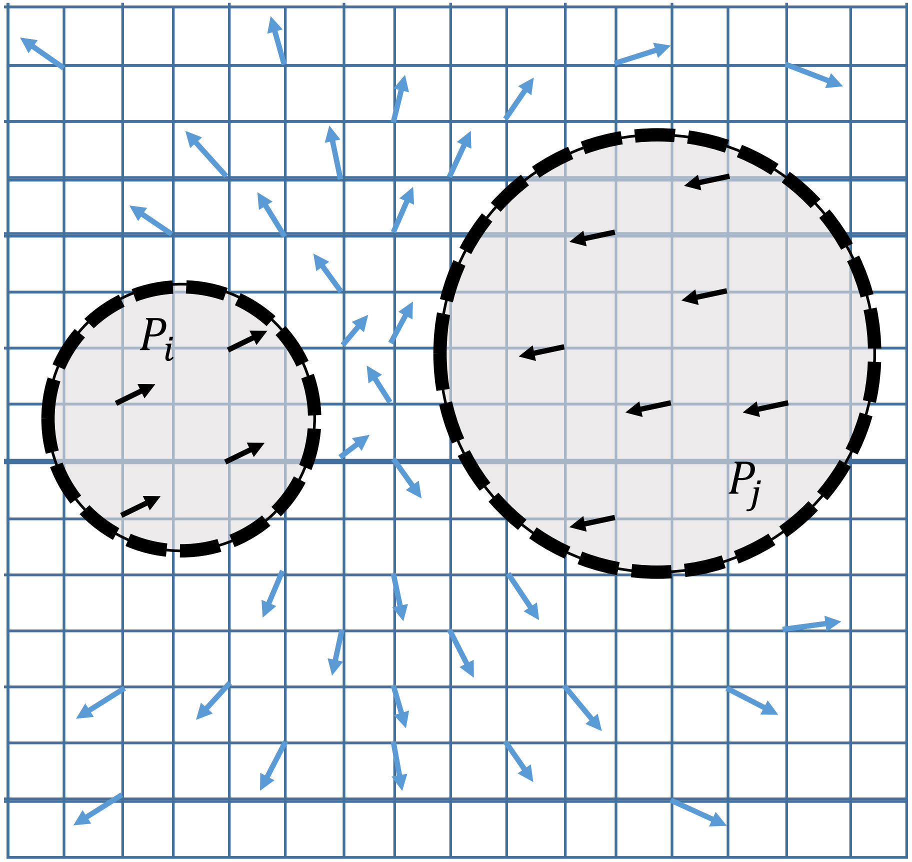
III. Some Numerical Results

Introduction

Definitions and Framework

The Local Lubrication Correction Model

Numerical Simulation: VP Method - DEM



Volume Penalization Method (VP):

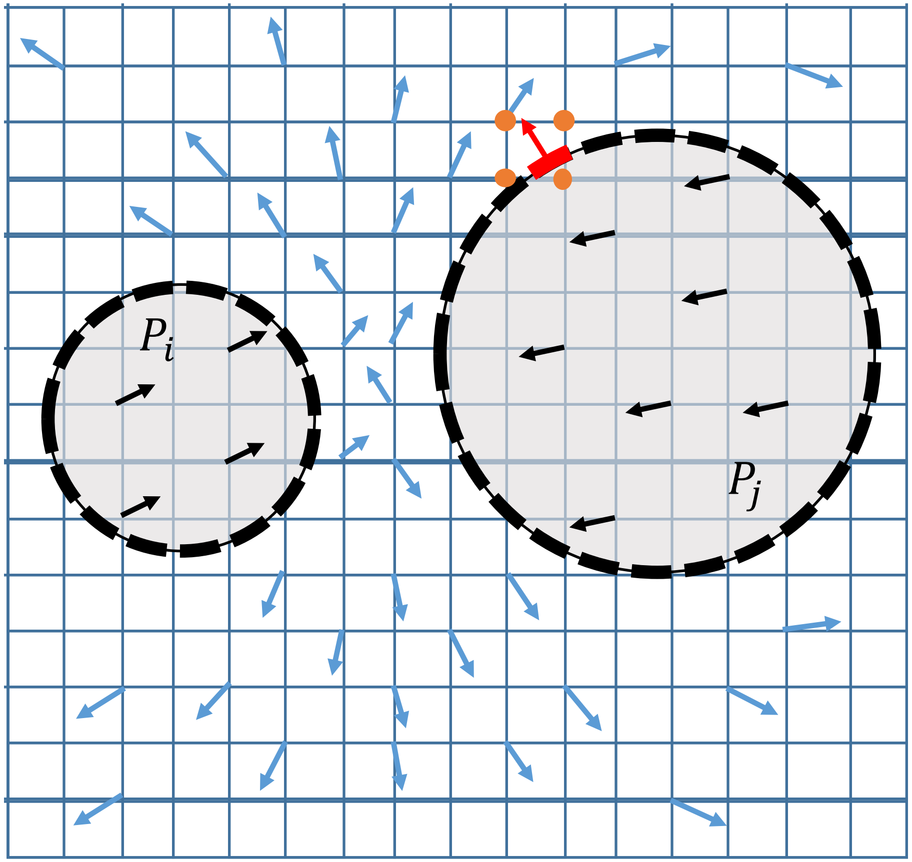
$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} &= -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\Delta \mathbf{u} + \frac{1}{\rho}\mathbf{F^{ext}} + \chi\lambda\,(\mathbf{u}_s - \mathbf{u}) \\ \nabla\cdot\mathbf{u} &= 0 \end{cases}$$

Particle dynamics:

$$m_i \frac{d\mathbf{U}_i}{dt} = \mathbf{F}_i^{hyd} + \mathbf{F}_i^{coll} + \mathbf{F}_i^{ext} + \mathbf{F}_i^{lub}$$

$$\overline{\overline{I}}_i \frac{d\Omega_i}{dt} = \mathbf{T}_i^{hyd} + \mathbf{T}_i^{coll} + \mathbf{T}_i^{lub}$$

Numerical Simulation: VP Method - DEM



Volume Penalization Method (VP):

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} &= -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\Delta \mathbf{u} + \frac{1}{\rho}\mathbf{F^{ext}} + \chi\lambda\,(\mathbf{u_s} - \mathbf{u}) \\ \nabla\cdot\mathbf{u} &= 0 \end{cases}$$

Particle dynamics:

$$m_i \frac{d\mathbf{U}_i}{dt} = \mathbf{F}_i^{hyd} + \mathbf{F}_i^{coll} + \mathbf{F}_i^{ext} + \mathbf{F}_i^{lub}$$

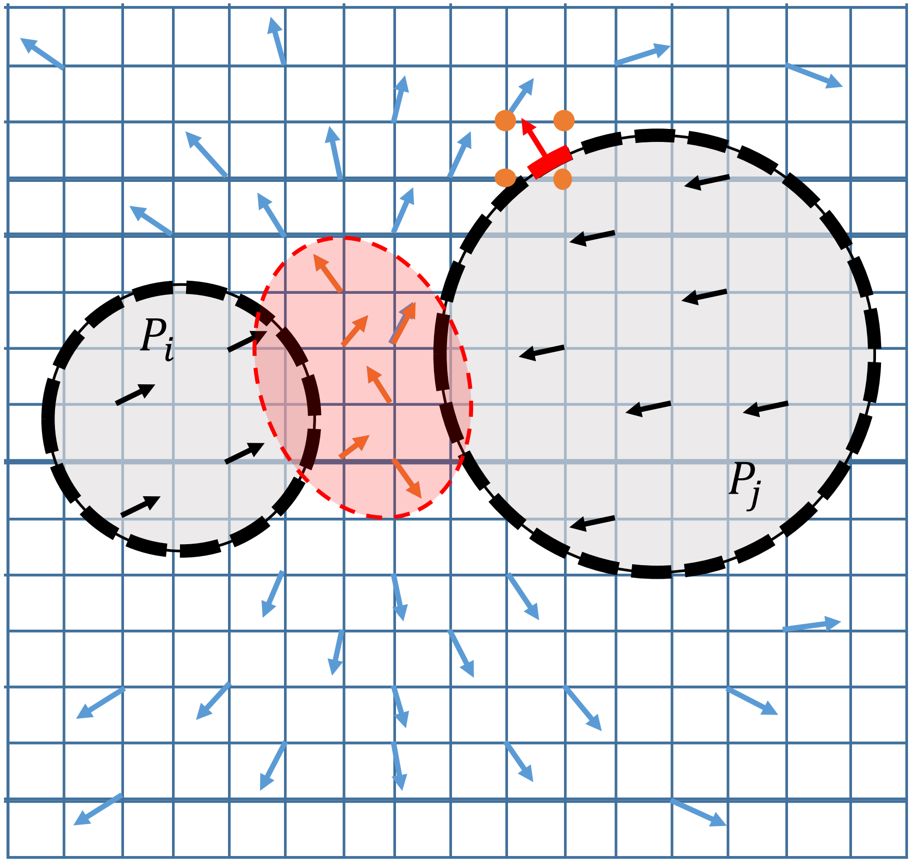
$$\overline{\overline{I}}_i \frac{d\mathbf{\Omega}_i}{dt} = \mathbf{T}_i^{hyd} + \mathbf{T}_i^{coll} + \mathbf{T}_i^{lub}$$

The hydrodynamic forces:

$$\mathbf{F}_i^{hyd} = \sum_{k\in\partial S_i} d\mathbf{f}_k^{hyd} = \sum_{k\in\partial S_i} \overline{\overline{\sigma}}\cdot\mathbf{n}_k$$

$$\overline{\overline{\sigma}} = -p\overline{\overline{I}} + \mu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})'\right)$$

Numerical Simulation: VP Method - DEM



Volume Penalization Method (VP):

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} &= -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\Delta \mathbf{u} + \frac{1}{\rho}\mathbf{F^{ext}} + \chi\lambda\,(\mathbf{u}_s - \mathbf{u}) \\ \nabla \cdot \mathbf{u} &= 0 \end{cases}$$

Particle dynamics:

$$m_i \frac{d\mathbf{U}_i}{dt} = \mathbf{F}_i^{hyd} + \mathbf{F}_i^{coll} + \mathbf{F}_i^{ext} + \mathbf{F}_i^{lub}$$

$$\overline{\overline{I}}_i \frac{d\boldsymbol{\Omega}_i}{dt} = \mathbf{T}_i^{hyd} + \mathbf{T}_i^{coll} + \mathbf{T}_i^{lub}$$

The hydrodynamic forces:

$$\mathbf{F}_i^{hyd} = \sum_{k \in \partial S_i} d\mathbf{f}_k^{hyd} = \sum_{k \in \partial S_i} \overline{\overline{\sigma}} \cdot \mathbf{n}_k$$

$$\overline{\overline{\sigma}} = -p\overline{\overline{I}} + \mu\left(\nabla \mathbf{u} + (\nabla \mathbf{u})'\right)$$

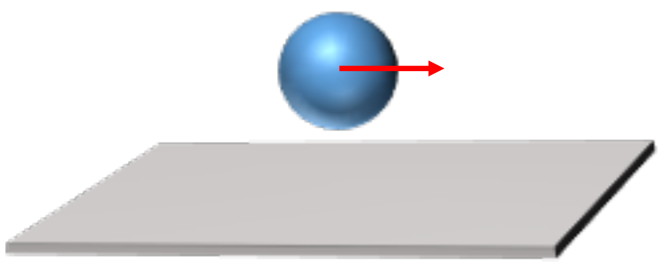
Lubrication Effects

Particle dynamics:

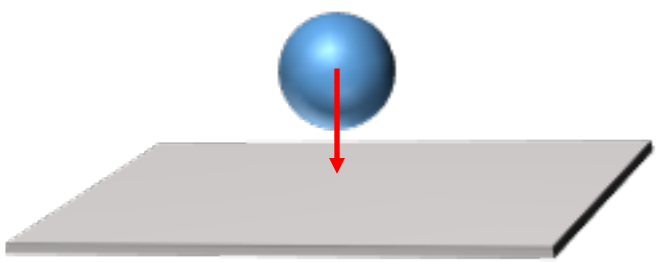
$$m_i \frac{d\mathbf{U}_i}{dt} = \mathbf{F}_i^{hyd} + \mathbf{F}_i^{coll} + \mathbf{F}_i^{ext} + \mathbf{F}_i^{lub}$$

$$\overline{\overline{I}}_i \frac{d\mathbf{\Omega}_i}{dt} = \mathbf{T}_i^{hyd} + \mathbf{T}_i^{coll} + \mathbf{T}_i^{lub}$$

Lubrication refers to the singular component of the hydrodynamic forces (and torques) induces by the presence of a nearby obstacle.



Shear motion



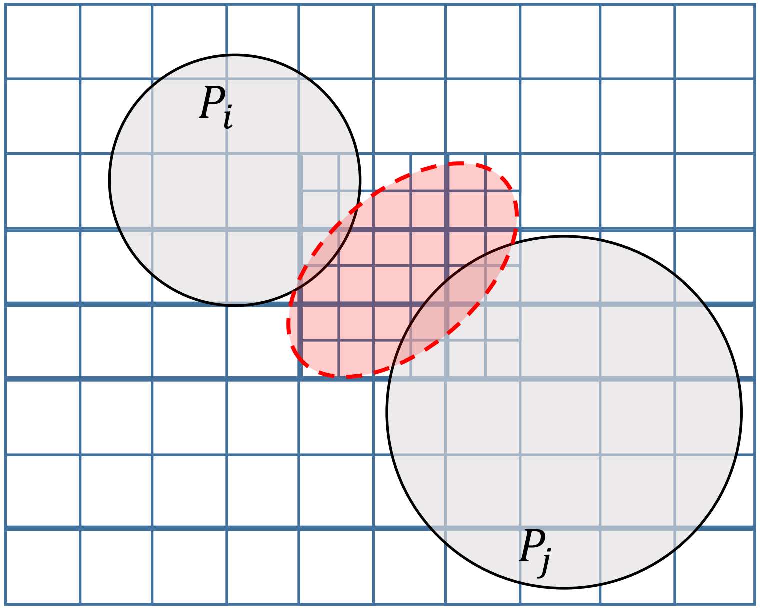
Squeeze motion

State of the Art

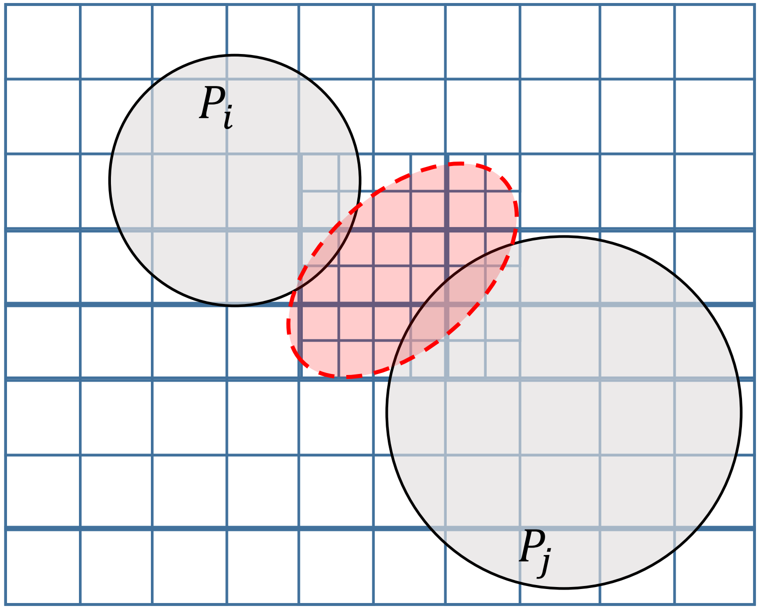
No lubrication correction:

→ Mesh refinement:

- No modelling
- Computation cost



State of the Art



No lubrication correction:

→ Mesh refinement:

- No modelling
- Computation cost

With lubrication correction:

→ Correction via tabulation:

Gallier *et. al.* (2013)

→ Correction via analytical solutions:

Izard *et. al.* (2014), Nguyen *et. al.* (2007), Jenkins (2005), …

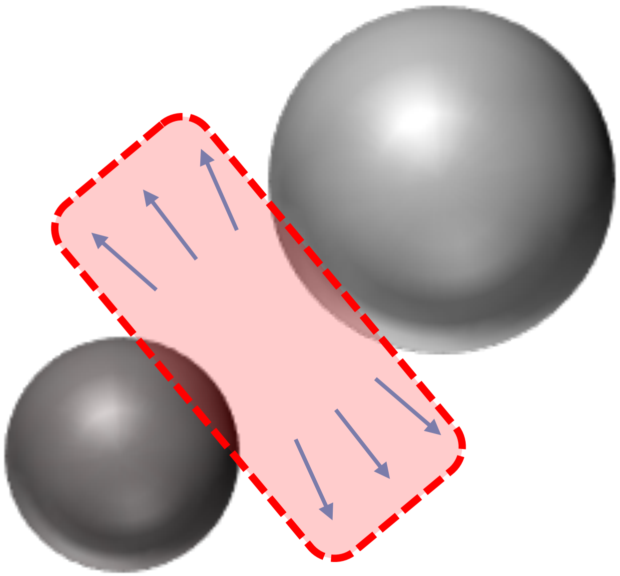
- Low computation cost
- Stokes regime and/or limited to spherical particles
- No correction of many body interactions

Definitions and Framework

The Local Lubrication Correction Model

Some Numerical Results

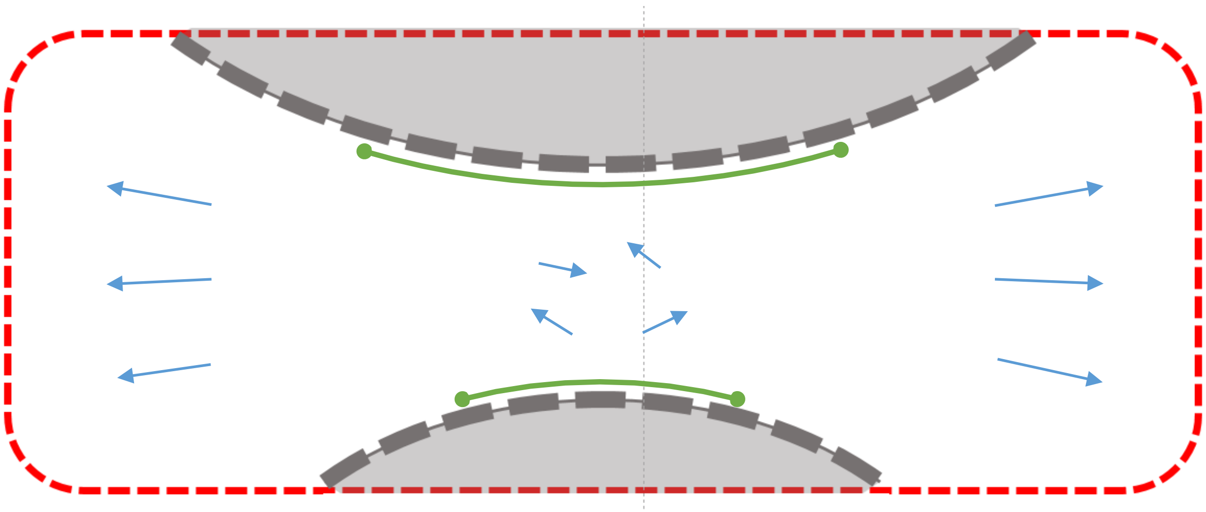
The LLCM: Objectives and Concept



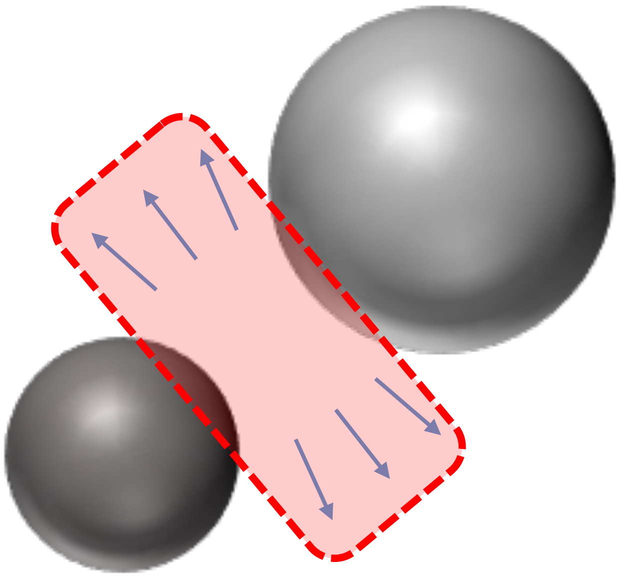
Objectives:

→ Compatible with Navier-Stokes flows

→ Particle geometry free



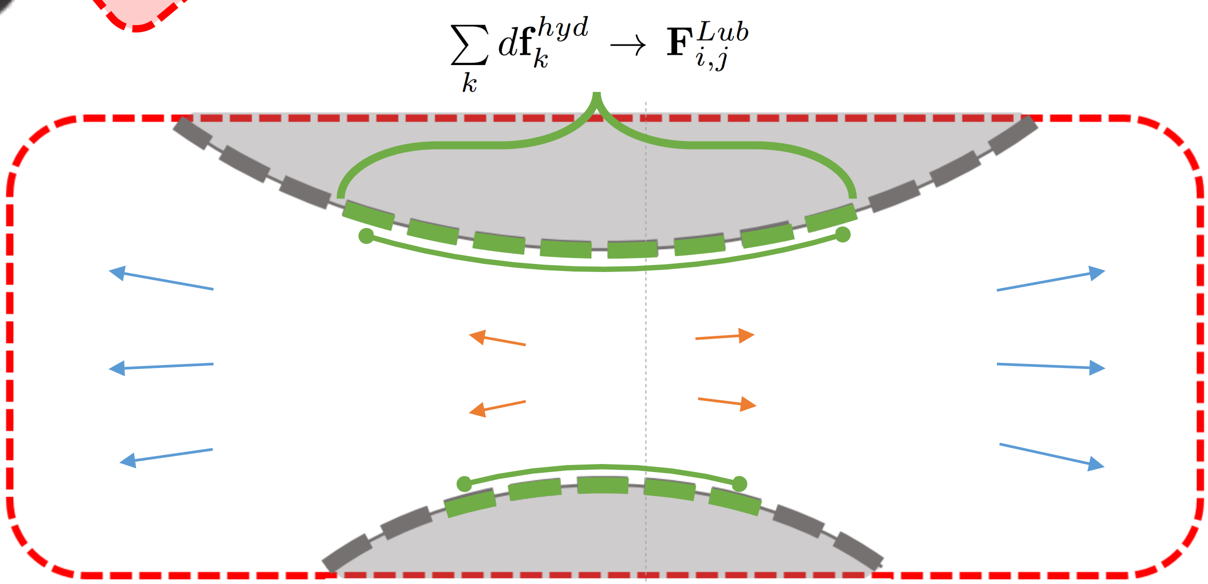
The LLCM: Objectives and Concept

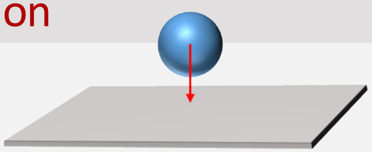


Objectives:

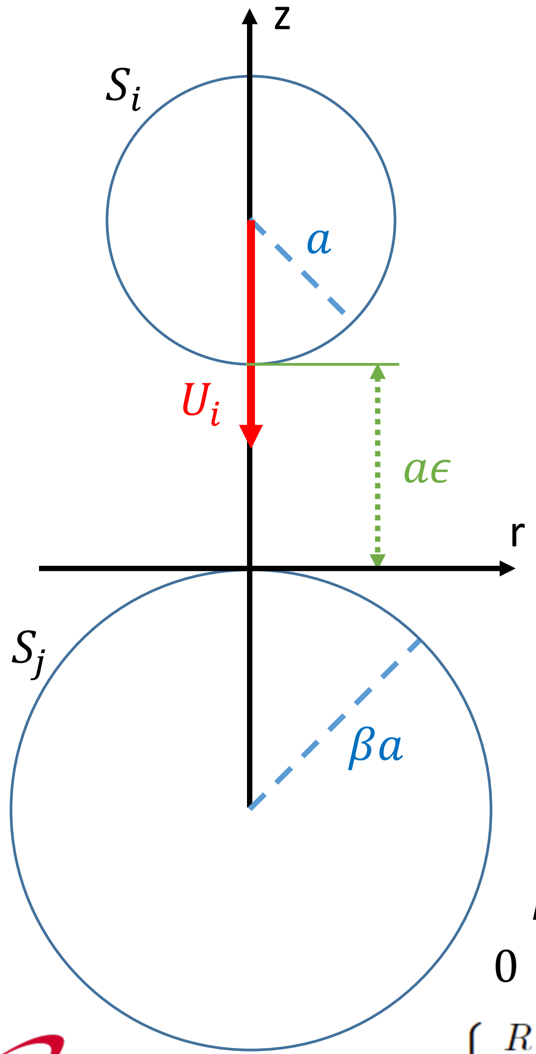
→ Compatible with Navier-Stokes flows

→ Particle geometry free





Squeezing Lubrication Force



$$\beta > 0$$

$$0 < \epsilon \ll 1$$

$$\begin{cases} R &= \frac{r}{a\sqrt{\epsilon}} \\ Z &= \frac{z}{a\epsilon} \end{cases}$$

Stokes flow:

$$\begin{cases} \mu \Delta \mathbf{u} &= \nabla p \\ \nabla \cdot \mathbf{u} &= 0 \\ \mathbf{u} \lceil_{\partial S_i} &= \mathbf{U} \lceil_{\partial S_i} \\ \mathbf{u} \lceil_{\partial S_j} &= \mathbf{U} \lceil_{\partial S_j} \\ \lim\limits_{r \to +\infty} p &= 0 \end{cases}$$

Axisymmetric flow:

$$(u_r, u_\theta, u_z) = \left(\tfrac{1}{r}\partial_z \psi, 0, -\tfrac{1}{r}\partial_r \psi\right)$$

Lubrication force:

$$\mathbf{F}_i^{lub} = \int_{\partial S_i} \sigma_{\mathbf{z}} \cdot \mathbf{i_n} dS$$

$$\mathbf{F}_i^{lub} = \pi \mu \int_C r^3 \partial_n \left(\frac{\Phi^2(\psi)}{r^2} \right) ds\, \mathbf{e_z}$$

Lubrication force in the “inner” region:

$$\frac{\mathbf{F}_i^{lub,in}}{\pi \mu a U_i} = -\frac{6R_0^4}{4H_0^2} \frac{1}{\epsilon} \mathbf{e_z} + O(1)$$

Reference:

Stimson et. al. (1926) Jeffrey (1982)

Summary: the LLCM

Leading order of forces and moment:

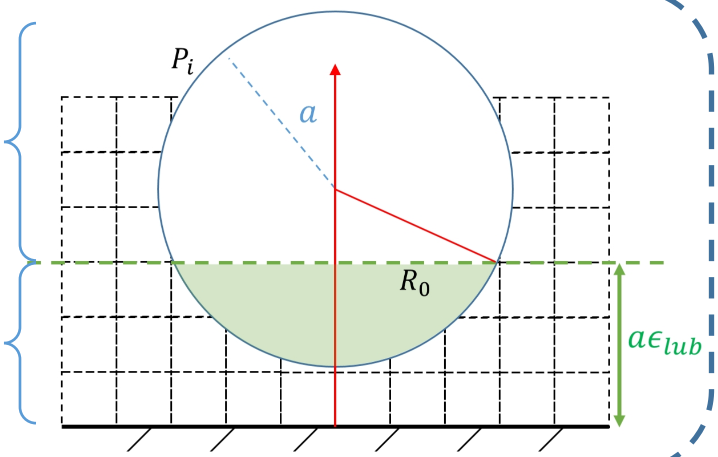
$$\frac{\mathbf{F}_i^{sq}}{\pi\mu a U_i^{sq}} = -\frac{6R_0^4}{4H_0^2}\frac{1}{\epsilon}\,\mathbf{e_z} + O(1)$$

$$\frac{\mathbf{F}_i^{sh}}{a\mu\pi U_i^{sh}} = \int_0^{R_0} \left[-P_0 R + \partial_Z V_0 - \partial_Z U_0\right] R dR\,\mathbf{e_x} + O(\epsilon)$$

$$\frac{\mathbf{T}_i^{sh}}{a^2\mu\pi U_i^{sh}} = \int_0^{R_0} \left[\partial_Z U_0 - \partial_Z V_0\right] R dR\,\mathbf{e_y} + O(\epsilon)$$

$$\mathbf{F}_i^{hyd} = \int_{\partial P_i} \overline{\overline{\sigma}} \cdot \mathbf{i_n} ds$$

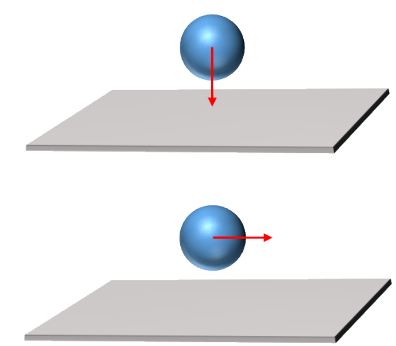
$$\mathbf{F}_i^{lub} = \mathbf{F}_i^{sq} + \mathbf{F}_i^{sh}$$



Main hypothesis:

→ Stokes regime within the gap

→ Flow and geometry symmetries

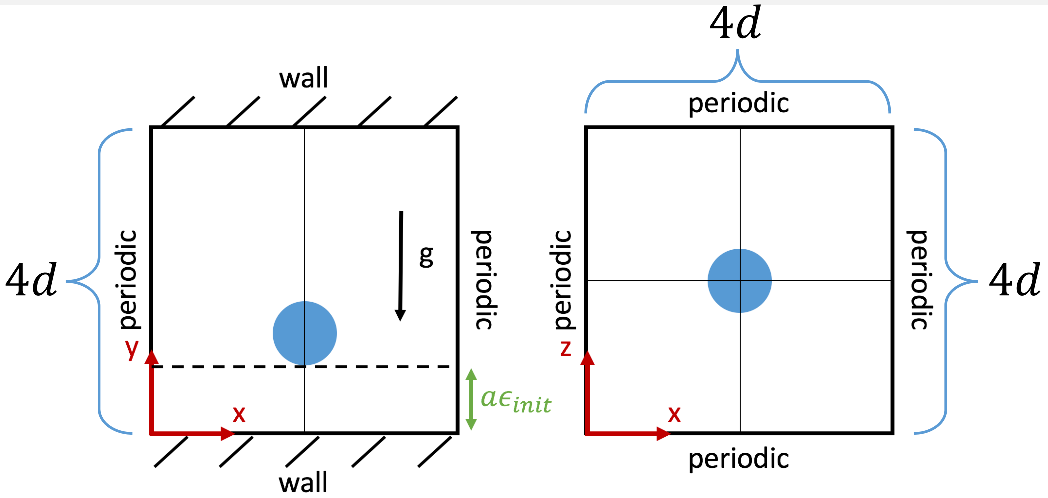


The Local Lubrication Correction Model

Some Numerical Results

Conclusion

Falling Particle: Description



$$\begin{cases} Re_d &= \frac{\rho U_T d}{\mu} \approx 25.7 \\ St_d &= \frac{\rho_p U_T d}{9\mu} \approx 3.27 \end{cases}$$

Fluid density	ρ	985	$kg.m^{-3}$
Fluid dynamic viscosity	μ	0.142	$Pa.s$
Particle density	ρ_p	1127	$kg.m^{-3}$
Particle Diameter	d	0.0254	m
Normal restitution	ξ_{max}	0.97	
Friction coefficient	μ_c	0.25	
Particle roughness	$a\epsilon_{col}$	$2.10^{-4}d$	m
Gravity field	g	9.781	$N.kg^{-1}$
Terminal Velocity	U_t	0.146	$m.s^{-1}$
Inital position	ϵ_{init}	0.4181	

Harada Model (Model H):

Lubrication force + Added-mass force + Basset history force

$$\mathbf{f}_l = 6\pi\mu a \frac{\mathbf{U}_i^{sq}}{\epsilon}$$

$$\begin{cases} \mathbf{f}_a &= m' \frac{d\mathbf{U}_i}{dt} + \frac{1}{2}\frac{dm'}{dt}\mathbf{U}_i \\ m' &= \frac{2}{3}\pi\rho a^3 \left(1 + \sum_{i=0}^{\infty} \frac{3a^{3(i+1)}}{f_0 f_1 \dots f_i}\right) \end{cases}$$

$$f_0 = 2a(\epsilon+1) \qquad f_i = f_0 - a^2/f_{i-1}.$$

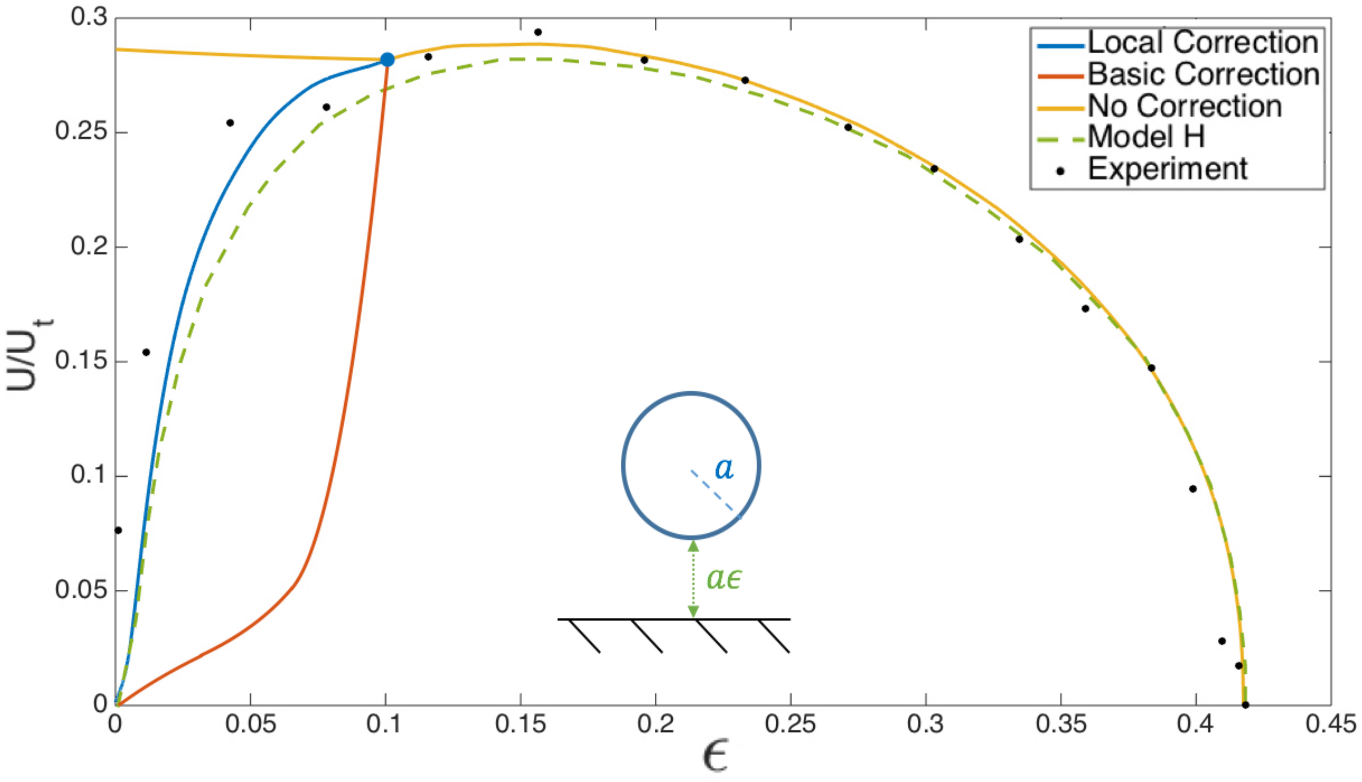
$$\mathbf{f}_{Ba} = 6a^2\sqrt{\pi\rho\mu} \int_{-\infty}^{t} \frac{d\mathbf{U}_i}{dt'} \frac{dt'}{\sqrt{t-t'}}$$

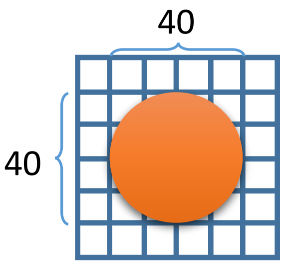
Basic Lubrication Correction Model (BLCM):

$$\frac{\mathbf{F}_i^{lub}}{\pi\mu a U_i} = -\frac{6}{\epsilon}\mathbf{e_z} + O(1)$$

Reference: Harada *et. al.* (2001)

Falling Particle: Velocity Profile



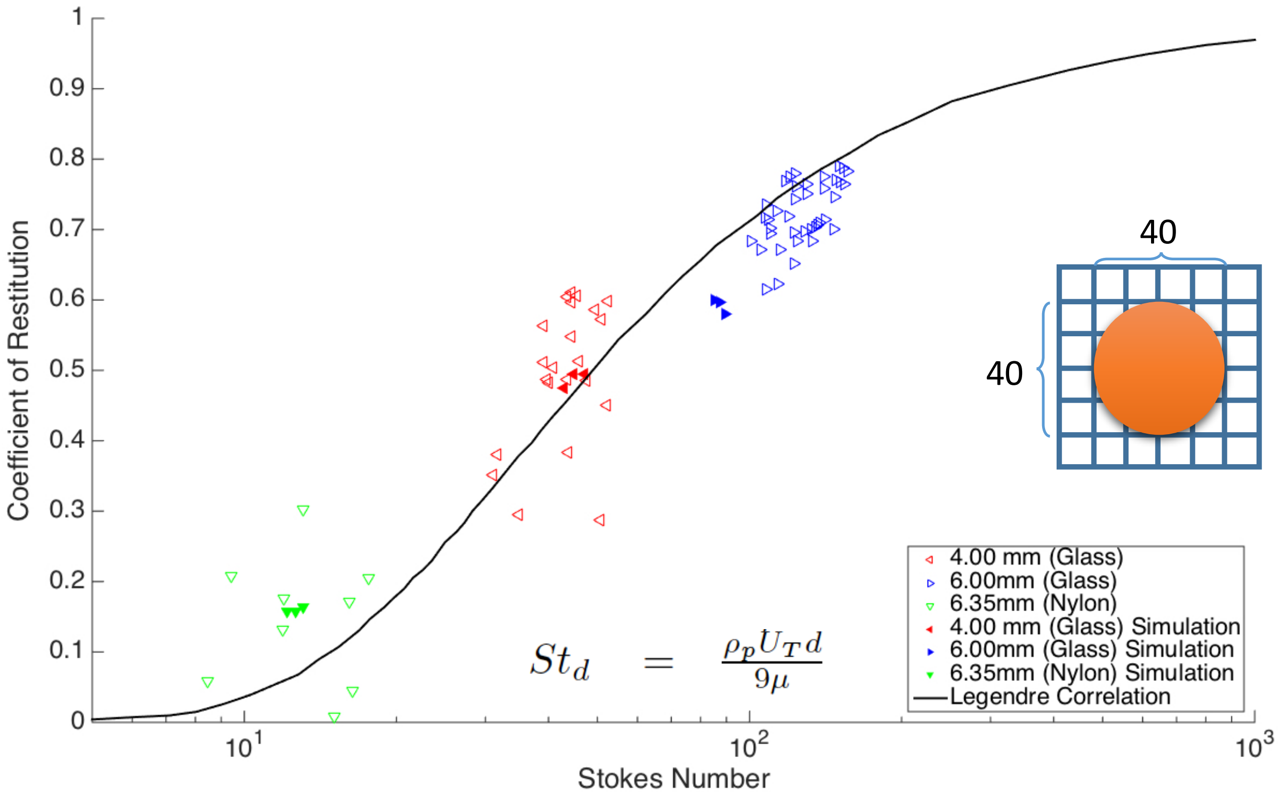


Reference: Harada *et. al.* (2001)

Falling Particle: Impact with rebound

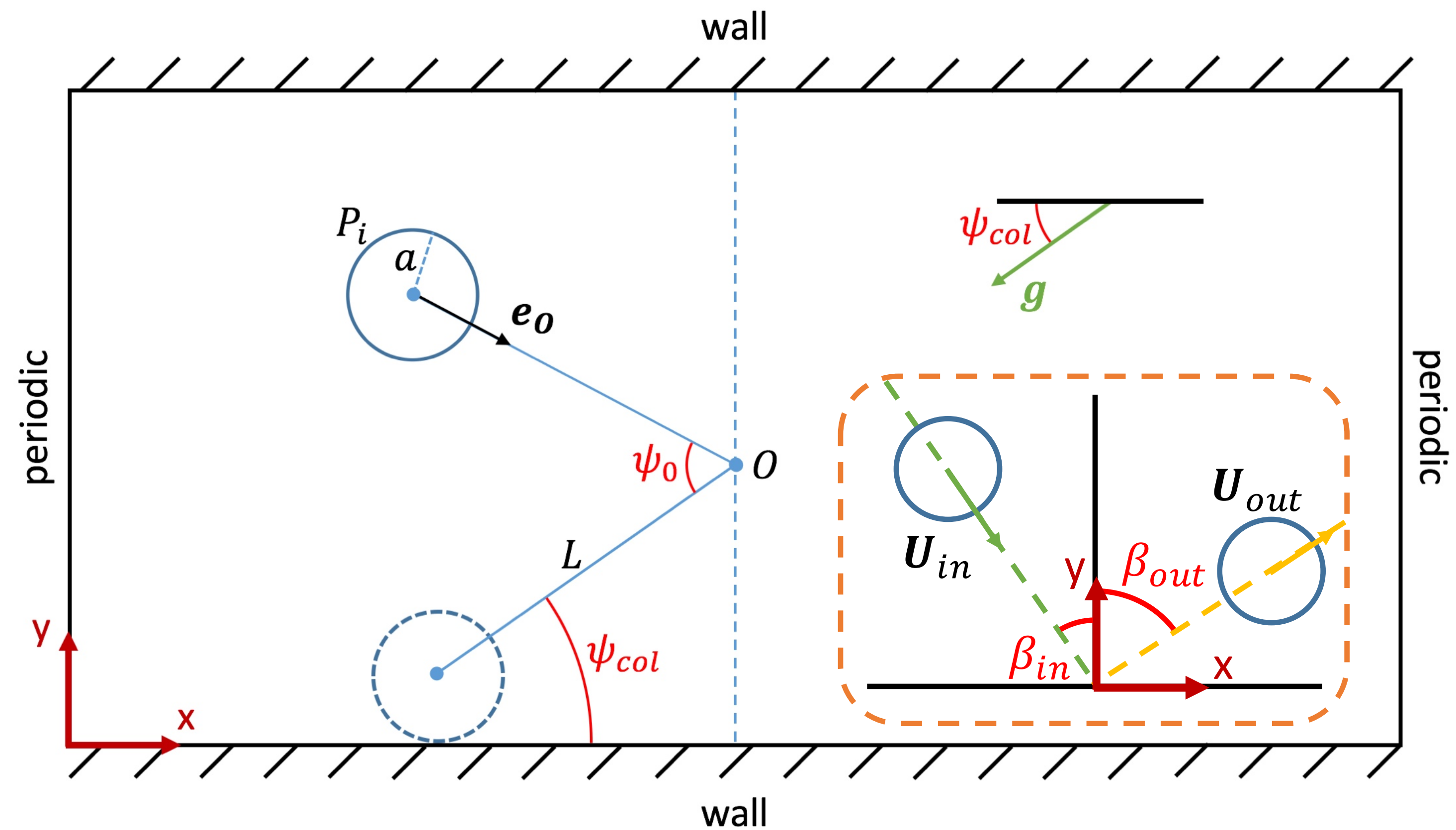
Normal coefficient of restitution:

$$\xi = -U_R/U_T$$



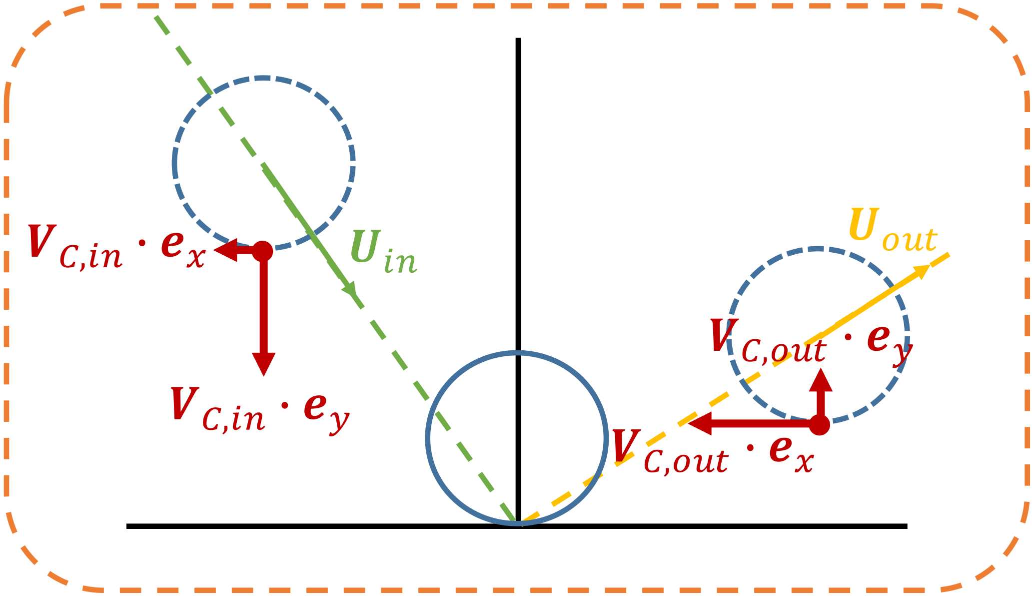
Reference: Joseph *et. al.* (2001)

Oblique Incidence: Description



Reference: Joseph *et. al.* (2004)

Oblique Incidence: Results



$$\Psi_{in} = -\frac{\mathbf{V}_{C,in} \cdot \mathbf{e_x}}{\mathbf{V}_{C,in} \cdot \mathbf{e_y}}$$

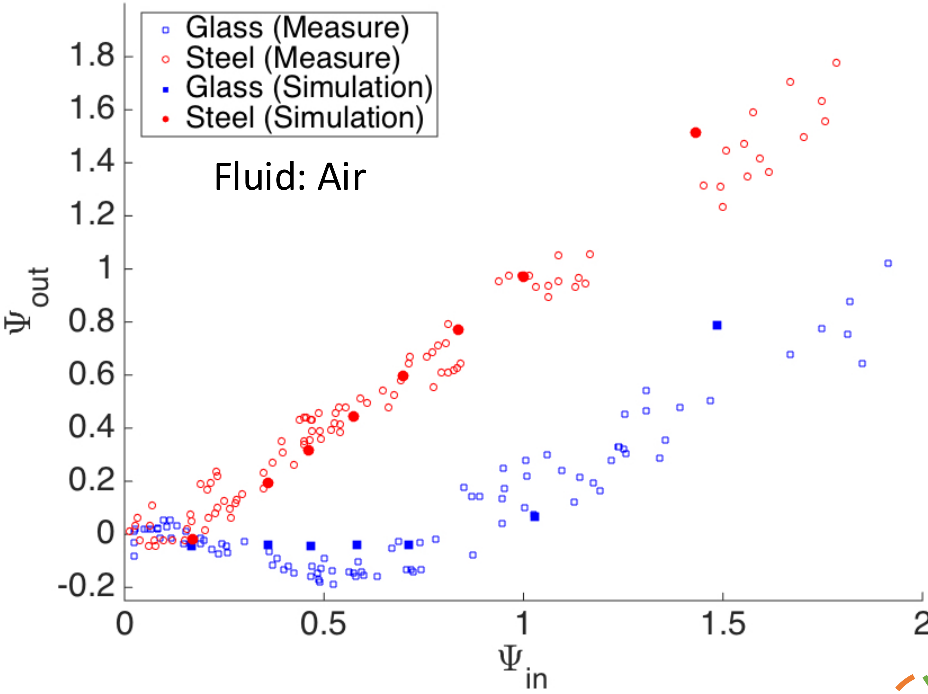
$$\Psi_{out} = \frac{\mathbf{V}_{C,out} \cdot \mathbf{e_x}}{\mathbf{V}_{C,out} \cdot \mathbf{e_y}}$$

$$\mathbf{V}_C = \mathbf{U}_i - a\,\mathbf{\Omega}_i \wedge \mathbf{e_y}$$

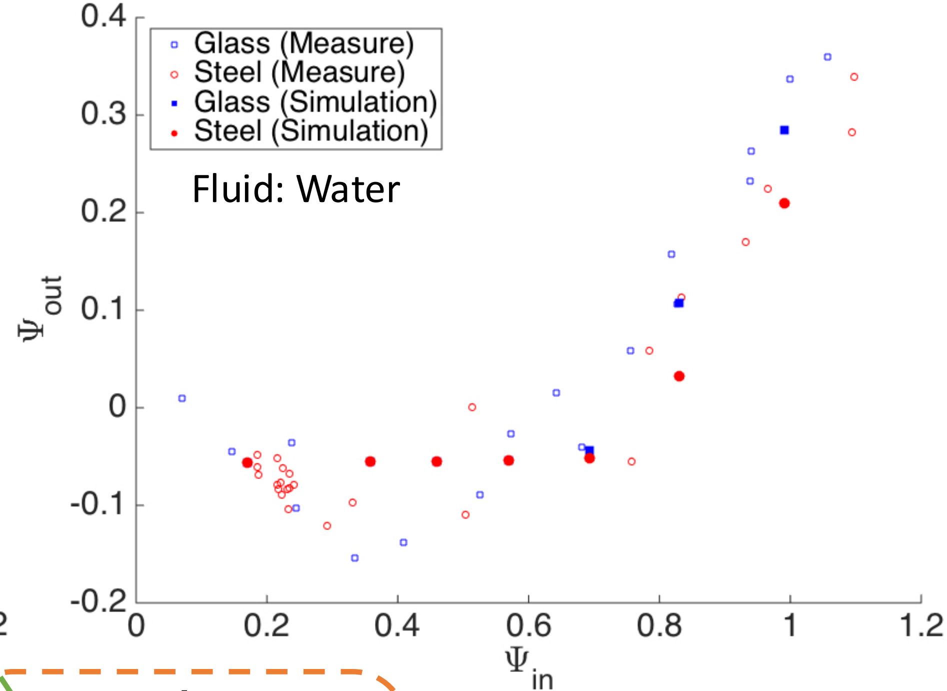
Reference: Joseph *et. al.* (2004)

Oblique Incidence: Results

Dry Collision

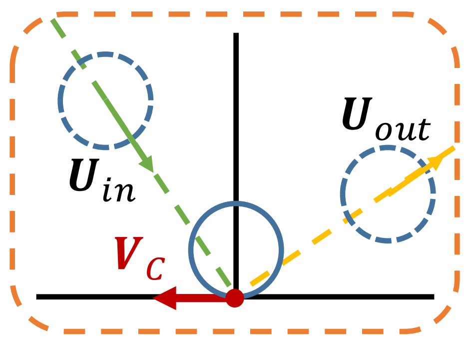


Wet Collision



$$\Psi_{in} = -\frac{\mathbf{V}_{C,in} \cdot \mathbf{e_x}}{\mathbf{V}_{C,in} \cdot \mathbf{e_y}}$$

$$\Psi_{out} = \frac{\mathbf{V}_{C,out} \cdot \mathbf{e_x}}{\mathbf{V}_{C,out} \cdot \mathbf{e_y}}$$



$$\mathbf{V}_C = \mathbf{U}_i - a\, \boldsymbol{\Omega}_i \wedge \mathbf{e_y}$$

Reference: Joseph *et. al.* (2004)

Institut de Mathématiques de Bordeaux

Inria INVENTORS FOR THE DIGITAL WORLD

Conclusion

Local Lubrication Correction Model:

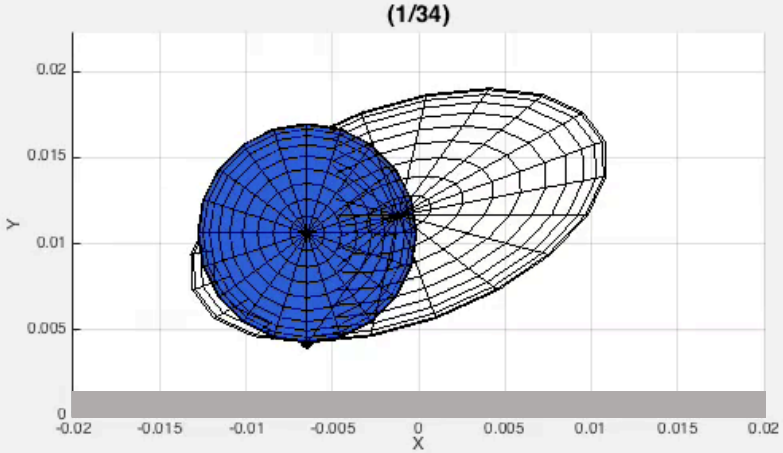
- Compatible with incompressible Navier Stokes flows
- Geometry independent correction

Limitations:

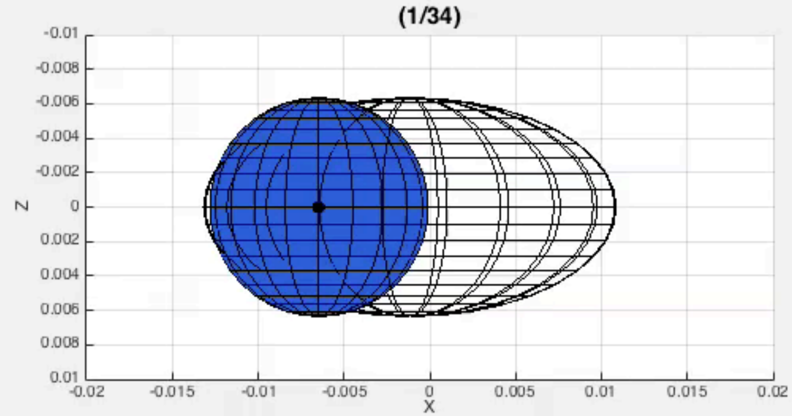
- No correction of the many body interactions
- No correction of fluid inertia in the gap

Outlooks

- Generalization of the method to non-spherical particles



Side View



From the ground

- Shear thickening of dense particle laden flow

Institut de Mathématiques de Bordeaux

INVENTORS FOR THE DIGITAL WORLD

Thank you for your attention

Any questions?

[1] S. Gallier, E. Lemaire, L. Lobry, F. Peters — *A Fictitious Domain Approach for the Simulation of Dense Suspensions.*
Journal of Computational Physics 01/2013; Volume 256: 367-387
(DOI: 10.1016/j.jcp.2013.09.015)

[2] E. Izard, T. Bonometti and L. Lacaze — *Modelling the Dynamics of a Sphere Approaching and Bouncing on a Wall in a Viscous Fluid.*
Journal of Fluid Mechanics, 747:422-446, 2014.

[3] Y. Nguyen, J. Wells and H. Truong — *A Fictitious-Domain Simulation of Solid-Liquid Flow with Subgrid Lubrication Force Correction; a Sphere Falling onto a Plane Surface.*
Proceedings of Hydraulic Engineering, 51:151-156, 2007.

[4] J. T. Jenkins and M. A. Koenders — *Hydrodynamic Interaction of Rough Spheres.*
Springer-Verlag - Granular Matter (2005) 7: 13–18
(DOI 10.1007/s10035-004-0193-7)

[5] M. Stimson, G. B. Jeffery — *The Motion of Two Spheres in a Viscous Fluid.*
The Royal Society 05/1926; 111(757).
(DOI: 10.1098/rspa.1926.0053)

[6] D. J. Jeffrey — *Low Reynolds Number Flow between Converging Spheres.*
Mathematika 05/1982; 29(01):58-66.
(DOI: 10.1112/S002557930001216X)

[7] S. Harada, T. Tanaka and Y. Tsuji Tsuji — *Fluid Force Acting on a Particle Falling Toward a Wall.*
JSME International Journal Series B 11/2001; 44(4): 520-525
(DOI: 10.1299/jsmeb.44.520)

[8] G. G. Joseph, M. L. Hunt, R. Zenit and A. M. Rosenwinkel — *Particle-Wall Wollisions in a Viscous Fluid.*
Journal of Fluid Mechanics, 433:329{346, 04/2001.

[9] G. G. Joseph and M. L. Hunt — *Oblique Particle-Wall Collisions in a Liquid.*
Journal of Fluid Mechanics 2004; Volume 510: 71-93
(DOI: 10.1017/S002211200400919X)

[10] M. E., O'Niell, K. Stewartson — *On the Slow Motion of a Sphere Parallel to a Nearby Plane Wall.*
Journal of Fluid Mechanics 03/1967; Volume 27(04): 705-724
(DOI: 10.1017/S0022112067002551)

[11] M. D. A. Cooley, M. E., O'Niell — *On the Slow Rotation of a Sphere about a Diameter Parallel to a Nearby Plane Wall.*
IMA Journal of Applied Mathematics 1968; 4(2): 163-173.
(DOI: 10.1093/ima-mat/4.2.163)

[12] S. Kim, S. Karrila — *Microhydrodynamics: Principles and Selected Applications.*
Dover Publications Inc. 1991
(ISBN 0-486-44219-5)

[13] M. N. Ardekani, P. Costa, W. P. Breugem, L. Brandt – *Numerical Study of the Sedimentation of Spheroidal Particles.*
02/2016; eprint arXiv:1602.05769

[14] P. Costa, B. J. Boersma, J. Westerweel and W. P. Breugem — *Collision Model for Fully-Resolved Simulations of Flows Laden with Finite-Size Particles.*
Physical Review E 10/2015; 92(5)
(DOI: 10.1103/PhysRevE.92.053012)

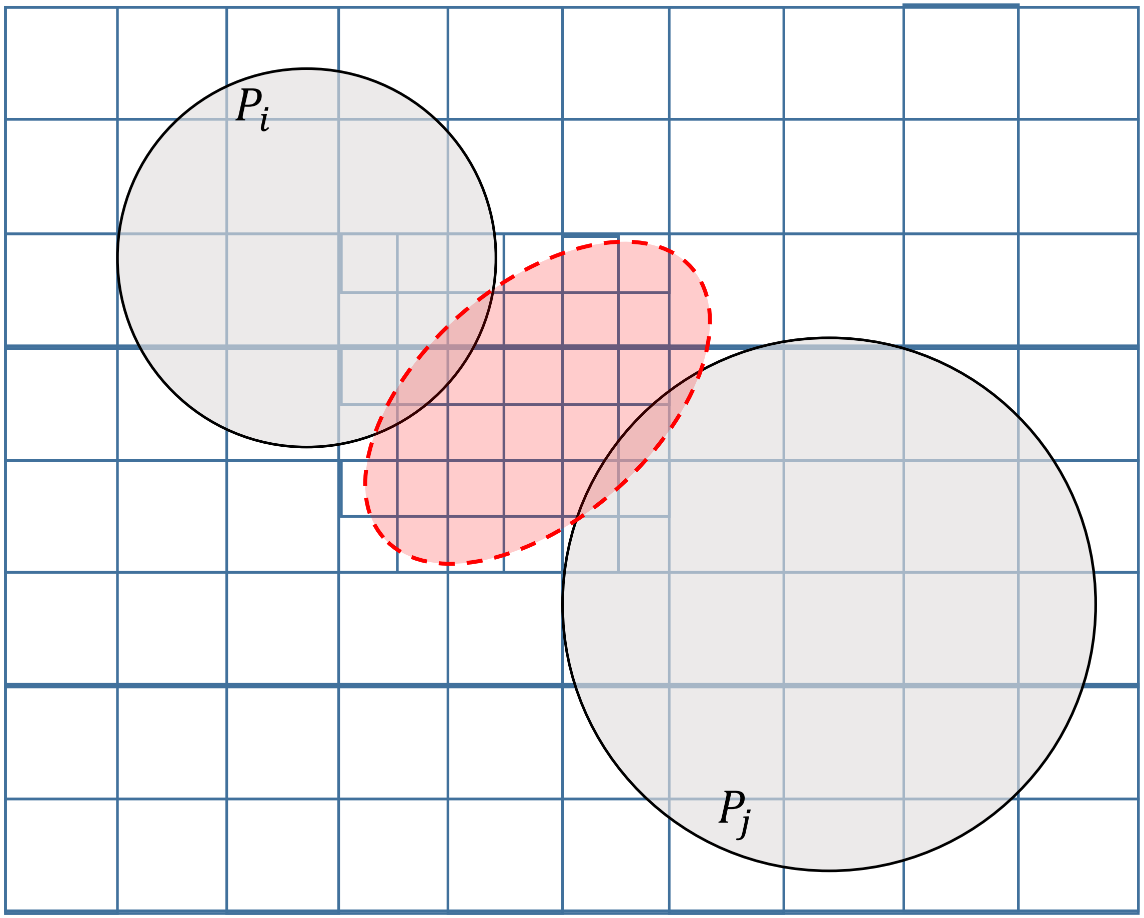
Annexes

Introduction

Definitions and Framework

The Local Lubrication Correction Model

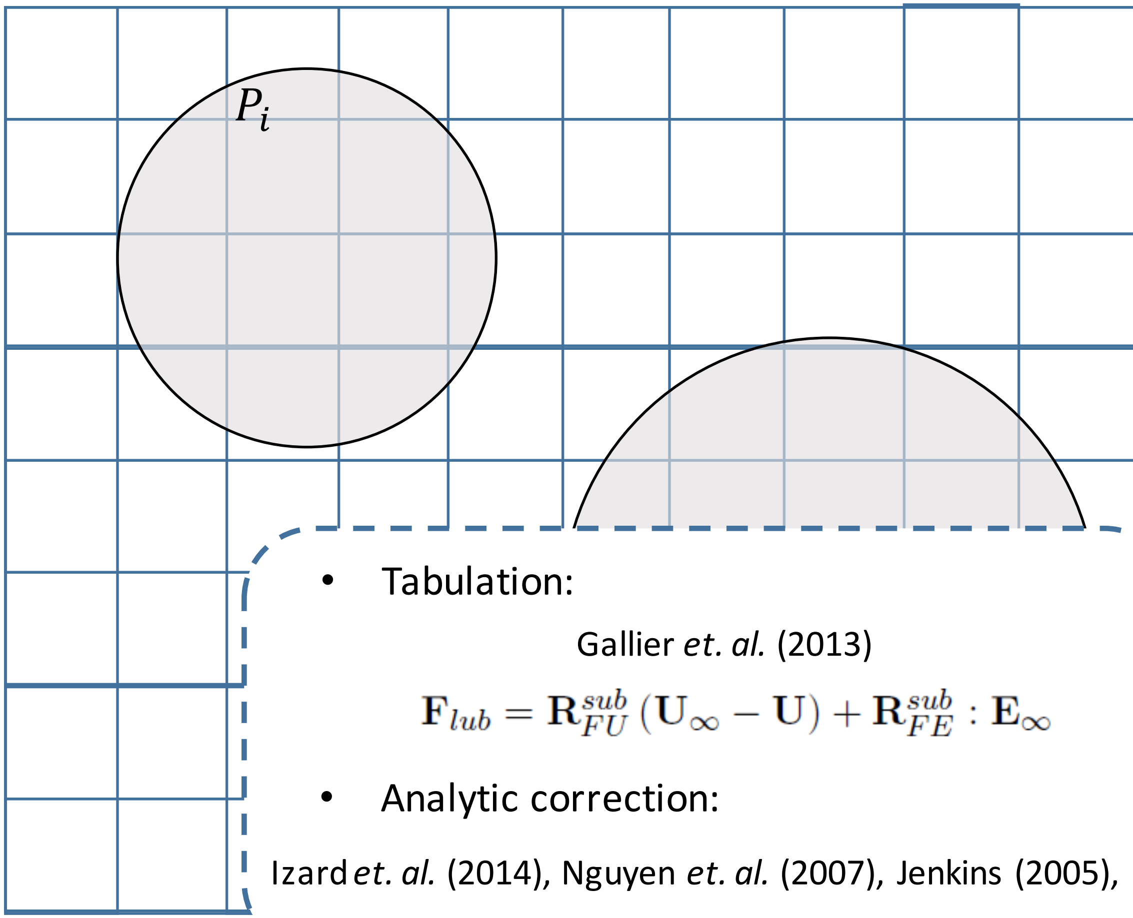
State of the Art



→ Mesh refinement:

- No modelling
- Computation cost

State of the Art



- Tabulation:

Gallier *et. al.* (2013)

$$\mathbf{F}_{lub} = \mathbf{R}_{FU}^{sub}\left(\mathbf{U}_{\infty} - \mathbf{U}\right) + \mathbf{R}_{FE}^{sub} : \mathbf{E}_{\infty}$$

- Analytic correction:

Izard *et. al.* (2014), Nguyen *et. al.* (2007), Jenkins (2005), …

→ Mesh refinement:

- No modelling
- Computation cost

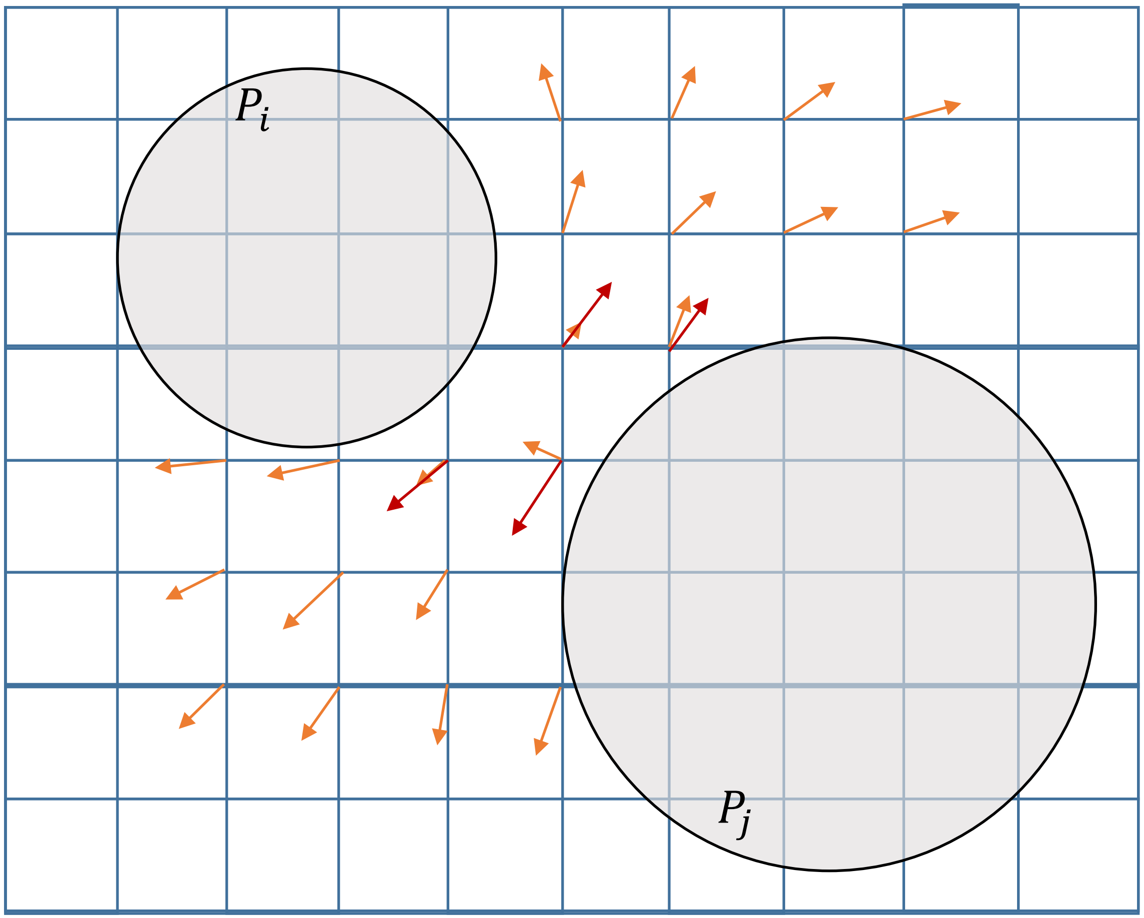
→ Correction of the hydrodynamic forces

- Low computation cost
- Lubrication model
- No many body interactions

Institut de Mathématiques de Bordeaux

INVENTORS FOR THE DIGITAL WORLD

State of the Art



→ Mesh refinement:

- No modelling
- Computation cost

→ Correction of the hydrodynamic forces

- Low computation cost
- Lubrication model
- No many body interactions

→ Correction of the flow

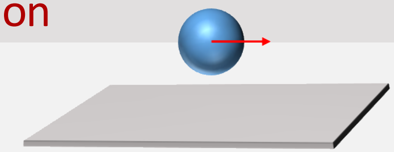
- Many body interactions
- No modelling
- Only for Stokes flow
- Off-line tabulation

Lefebvre-Lepot *et. al.* (2015)

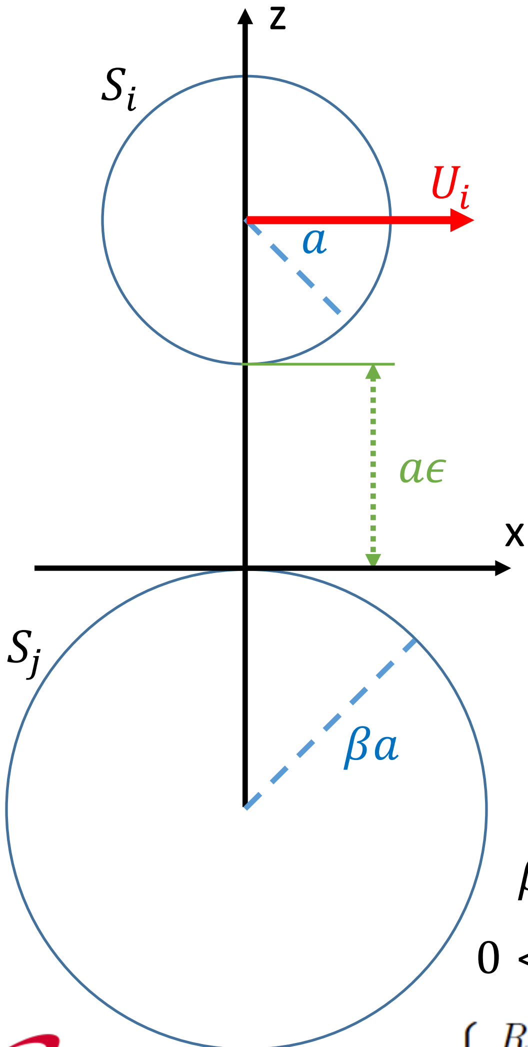
Definitions and Framework

The Local Lubrication Correction Model

Some Numerical Results



Shearing Lubrication Force and Moment



$\beta > 0$

$0 < \varepsilon \ll 1$

$$\begin{cases} R = \frac{r}{a\sqrt{\epsilon}} \\ Z = \frac{z}{a\epsilon} \end{cases}$$

Stokes flow:

$$\begin{cases} \mu \Delta \mathbf{u} = \nabla p \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u} \restriction_{\partial S_i} = \mathbf{U} \restriction_{\partial S_i} \\ \mathbf{u} \restriction_{\partial S_j} = \mathbf{U} \restriction_{\partial S_j} \\ \lim_{r \to +\infty} p = 0 \end{cases}$$

Solution:

$$\begin{cases} u_r = U_i U(r,z) \cos(\theta) \\ u_\theta = U_i V(r,z) \sin(\theta) \\ u_z = U_i W(r,z) \cos(\theta) \\ p = \frac{\mu U_i}{a} P(r,z) \cos(\theta) \end{cases}$$

Lubrication force and moment:

$$\frac{F_x}{a\mu\pi U_A} = \int_0^{R_0} \left[-P_0 R + \partial_Z V_0 - \partial_Z U_0\right] R dR + O(\epsilon)$$

$$\frac{T_y}{a^2\mu\pi U_A} = \int_0^{R_0} \left[\partial_Z U_0 - \partial_Z V_0\right] R dR + O(\epsilon)$$

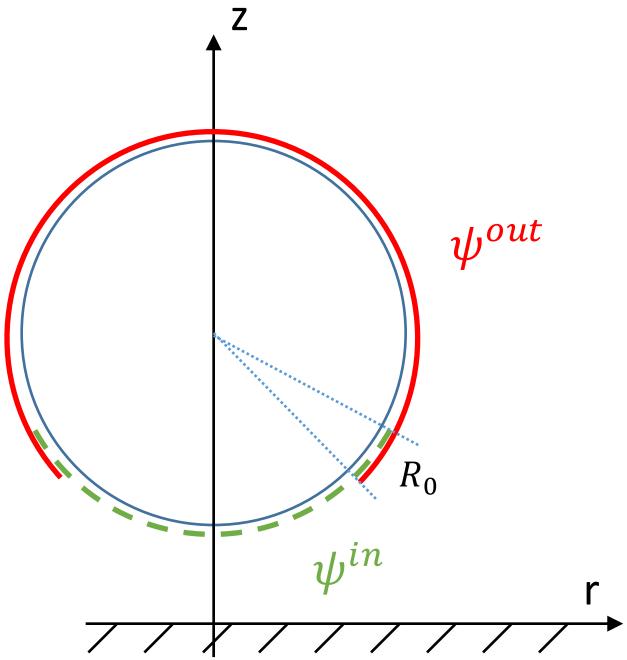
$$\begin{cases} P_0(R) = \frac{6R}{5H^2}\left(1 - \frac{1}{\beta}\right) \\ U_0(R,Z) = -\frac{1}{2}\frac{dP_0}{dR}(Z - Z_j)(Z_i - Z) + \frac{Z - Z_j}{H} \\ V_0(R,Z) = -\frac{1}{2}\frac{P_0}{R}(Z - Z_j)(Z_i - Z) - \frac{Z - Z_j}{H} \end{cases}$$

Reference:

O'Neill (1967) Cooley et. al. (1968)

Inner and Outer Solution Matching

How to characterized the lubrication region? (R_0?)



Two options:

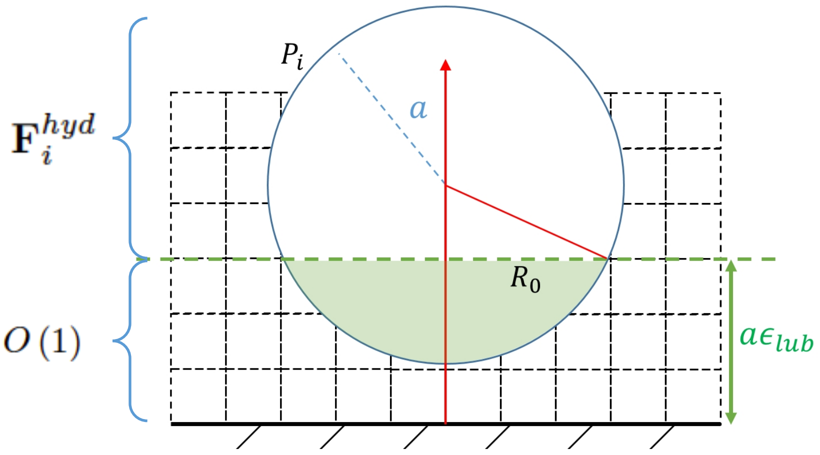
→ Asymptotic analysis

See: O'Neill *et. al.* (1967)

→ Curve fitting

Lubrication force on the particle surface:

$$\frac{\mathbf{F}_i^{lub,in}}{\pi\mu a U_i} = -\frac{6R_0^4}{4H_0^2}\frac{1}{\epsilon}\mathbf{e_z} + O(1)$$



Reference: Kim *et. al.* (1991)

The Local Lubrication Correction Model

Collision Model

Some Numerical Results

Introduction

Squeeze lubrication forces:

$$O\left(\frac{1}{\epsilon}\right)$$

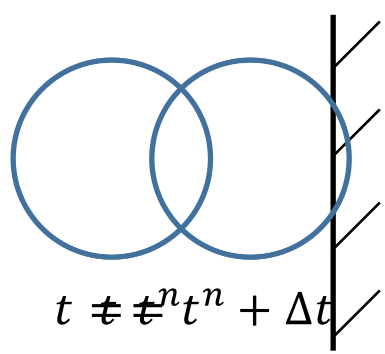
$$\frac{\mathbf{F}_i^{lub}}{\pi\mu a U_i} = -\frac{6}{\epsilon}\mathbf{e_z} + O(1)$$

In Stokes regime, collision are impossible

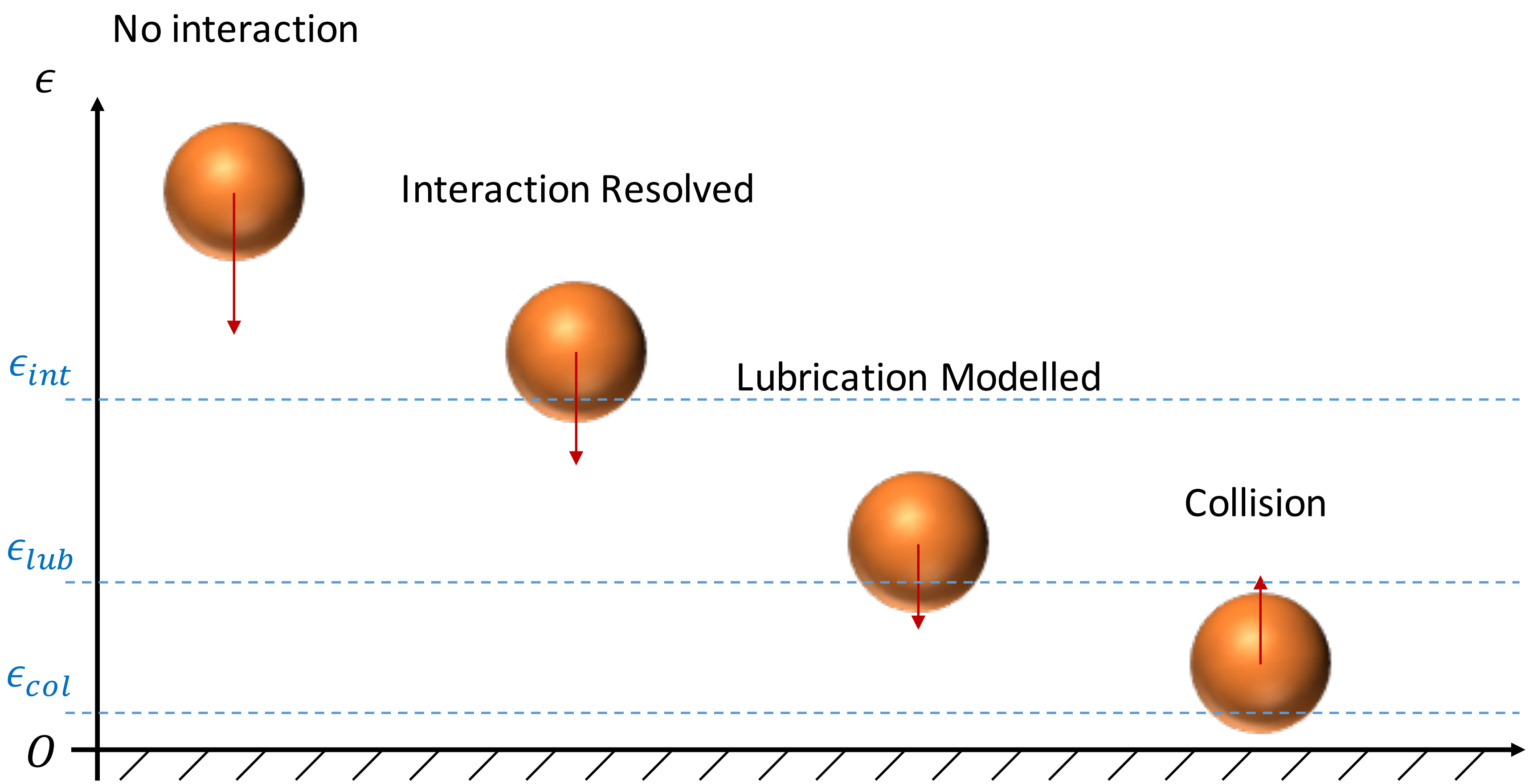
However:

→ Need to model particle deformations

→ Collision model insure numerical stability

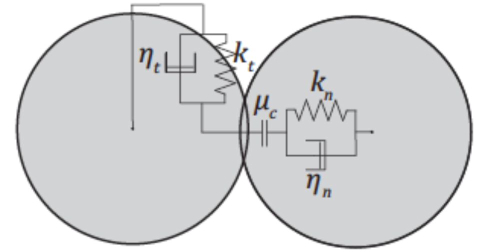


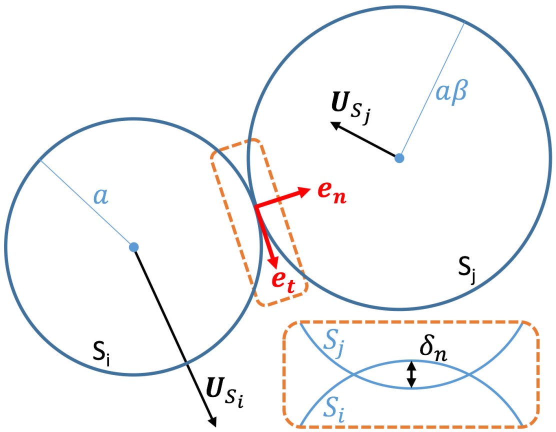
Particle-Wall Interaction Sequence



The Soft Sphere Contact Model

Linear spring-dashpot system with Coulomb-like threshold





$$\begin{cases} \mathbf{F}_{S_iS_j} &= \mathbf{F}_n + \mathbf{F}_t \\ \mathbf{T}_{S_iS_j} &= a\mathbf{e_n} \wedge (\mathbf{F}_t) \end{cases}$$

$$\begin{cases} \mathbf{F}_n &= -\boldsymbol{\delta_n} k_n - \gamma_n \left(\mathbf{U}_{i,j} \cdot \mathbf{e_n}\right) \mathbf{e_n} \\ \mathbf{F}_t &= \min\left(\| -\boldsymbol{\delta_t} k_t - \gamma_t \left(\mathbf{U}_{i,j} \cdot \mathbf{e_t}\right) \mathbf{e_t} \|, \|\mu_c \mathbf{F}_n\|\right) \mathbf{e_t} \end{cases}$$

Distances of overlap:

$$\boldsymbol{\delta_n} = \max\left(0, a\left(1+\beta\right) + \epsilon_{col}\left(a+\beta a\right) - \|\mathbf{X}_{S_i} - \mathbf{X}_{S_j}\|\right) \mathbf{e_n}$$

$$\boldsymbol{\delta_t}^{n+1} = \begin{cases} \overline{\overline{R}} \cdot \boldsymbol{\delta_t}^n + \int_{t^n}^{t^{n+1}} \left(\mathbf{U}_{i,j} \cdot \mathbf{e_t}\right) \mathbf{e_t} dt & , \|\mathbf{F}_t\| \leq \mu_c \|\mathbf{F}_n\| \\ (1/k_t)\left(-\mu_c \|\mathbf{F}_n\| \mathbf{e_t} - \gamma_t \left(\mathbf{U}_{i,j} \cdot \mathbf{e_t}\right) \mathbf{e_t}\right) & , \|\mathbf{F}_t\| > \mu_c \|\mathbf{F}_n\| \end{cases}$$

Stiffness and damping coefficient:

$$\begin{cases} k_n = \dfrac{m^*\left(\pi^2 + \ln^2(\xi_{max,n})\right)}{t_c^2} \\ k_t = \dfrac{m_t^*\left(\pi^2 + \ln^2(\xi_{max,t})\right)}{t_c^2} \end{cases} \qquad \begin{matrix} \gamma_n = -\dfrac{2m^* \ln(\xi_{max,n})}{t_c} \\ \gamma_t = -\dfrac{2m_t^* \ln(\xi_{max,t})}{t_c} \end{matrix}$$

$$\begin{cases} \mathbf{F}_i^{col} &= \sum\limits_{S \neq S_i} \mathbf{F}_{S_iS} + \mathbf{F}_{Wall} \\ \mathbf{T}_i^{col} &= \sum\limits_{S \neq S_i} \mathbf{T}_{S_iS} + \mathbf{T}_{Wall} \end{cases}$$

Institut de Mathématiques de Bordeaux

Reference: Costa *et. al.* (2015)

Inria — INVENTORS FOR THE DIGITAL WORLD

Collision Model

Numerical Resolution

Some numerical results

Scalar Projection Method

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} &= -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\Delta \mathbf{u} + \frac{1}{\rho}\mathbf{F}^{\mathbf{ext}} + \chi\lambda\,(\mathbf{u}_s - \mathbf{u}) \\ \nabla\cdot\mathbf{u} &= 0 \end{cases}$$

$$\mathbf{u}^* = \mathbf{u}^n - \Delta t\left(\left[(\mathbf{u}\nabla)\,\mathbf{u}\right]^{n+1/2} - \nu\Delta\mathbf{u}^{n+1/2} - \frac{1}{\rho}\nabla q\right)$$

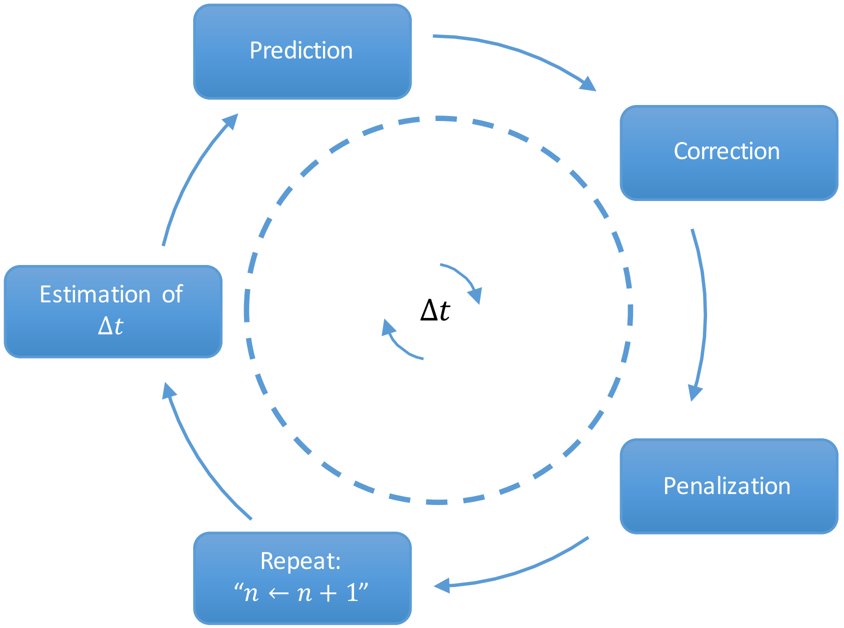
$$\left[(\mathbf{u}\nabla)\,\mathbf{u}\right]^{n+1/2} \approx \frac{3}{2}\left[\mathbf{u}\cdot\nabla\mathbf{u}\right]^n - \frac{1}{2}\left[\mathbf{u}\cdot\nabla\mathbf{u}\right]^{n-1}$$

$$\nabla\cdot\mathbf{u}^* = \Delta t\nabla^2\Phi$$

$$\Phi = \frac{p^{n+1} - q}{\rho}$$

$$\begin{cases} p^{n+1} &= \rho\Phi + p^n \\ \tilde{\mathbf{u}} &= \mathbf{u}^* - \Delta t\nabla\Phi \end{cases}$$

$$\Delta t \leq \frac{\Delta x}{V_{max}}$$

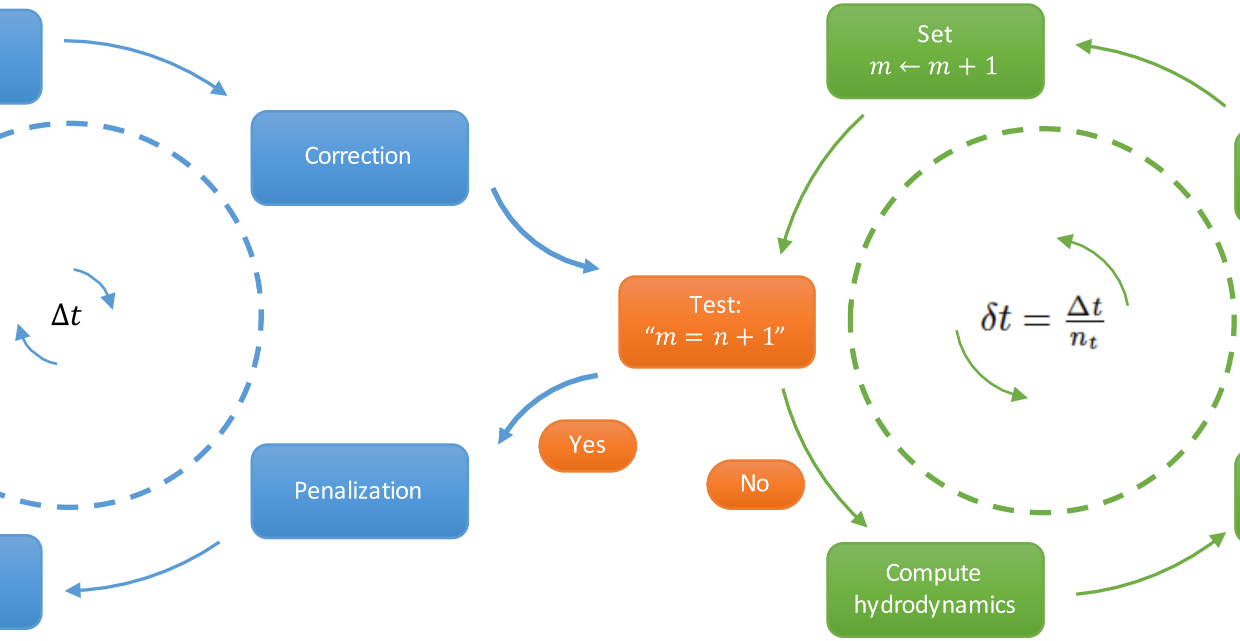


$$\mathbf{u}^{n+1} = \frac{\tilde{\mathbf{u}} + \chi\lambda\Delta t\mathbf{u}_\tau}{1 + \chi\lambda\Delta t}.$$

Particle Dynamics

$$m_i \frac{d\mathbf{U}_i}{dt} = \mathbf{F}_i^{hyd} + \mathbf{F}_i^{coll} + \mathbf{F}_i^{ext} + \mathbf{F}_i^{lub}$$

$$\overline{\overline{I}}_i \frac{d\Omega_i}{dt} = \mathbf{T}_i^{hyd} + \mathbf{T}_i^{coll} + \mathbf{T}_i^{lub}$$



$$\mathbf{U}_i^{m+1} = \mathbf{U}_i^m + \frac{\delta t}{m_i}\left(\left[\mathbf{F}_i^{hyd}\right]^{n+1} + \left[\mathbf{F}_i^{coll} + \mathbf{F}_i^{ext} + \mathbf{F}_i^{lub}\right]^m\right)$$

$$\Omega_i^{m+1} = \Omega_i^m + \delta t \left(\overline{\overline{I}}_i\right)^{-1}\left(\left[\mathbf{T}_i^{hyd}\right]^{n+1} + \left[\mathbf{T}_i^{coll} + \mathbf{T}_i^{lub}\right]^m\right)$$

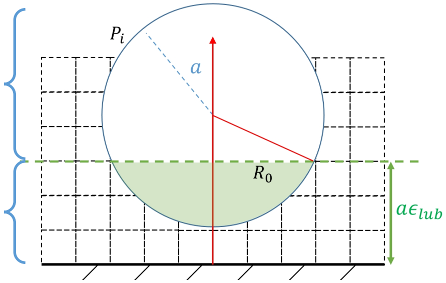
$$\begin{cases} \mathbf{F}_i^{col} &= \sum_{S \neq S_i} \mathbf{F}_{S_i S} + \mathbf{F}_{Wall} \\ \mathbf{T}_i^{col} &= \sum_{S \neq S_i} \mathbf{T}_{S_i S} + \mathbf{T}_{Wall} \end{cases}$$

$$\begin{cases} \mathbf{F}_{S_i S_j} &= \mathbf{F}_n + \mathbf{F}_t \\ \mathbf{T}_{S_i S_j} &= a\mathbf{e_n} \wedge (\mathbf{F}_t) \end{cases}$$

$$\begin{cases} \mathbf{F}_n &= -\boldsymbol{\delta_n} k_n - \gamma_n (\mathbf{U}_{i,j} \cdot \mathbf{e_n}) \mathbf{e_n} \\ \mathbf{F}_t &= \min\left(\| - \boldsymbol{\delta_t} k_t - \gamma_t (\mathbf{U}_{i,j} \cdot \mathbf{e_t}) \mathbf{e_t} \|, \|\mu_c \mathbf{F}_n\|\right) \mathbf{e_t} \end{cases}$$

$$\mathbf{F}_i^{hyd} = \int_{\partial P_i} \overline{\overline{\sigma}} \cdot \mathbf{i_n} ds$$

$$\mathbf{F}_i^{lub} = \mathbf{F}_i^{sq} + \mathbf{F}_i^{sh}$$

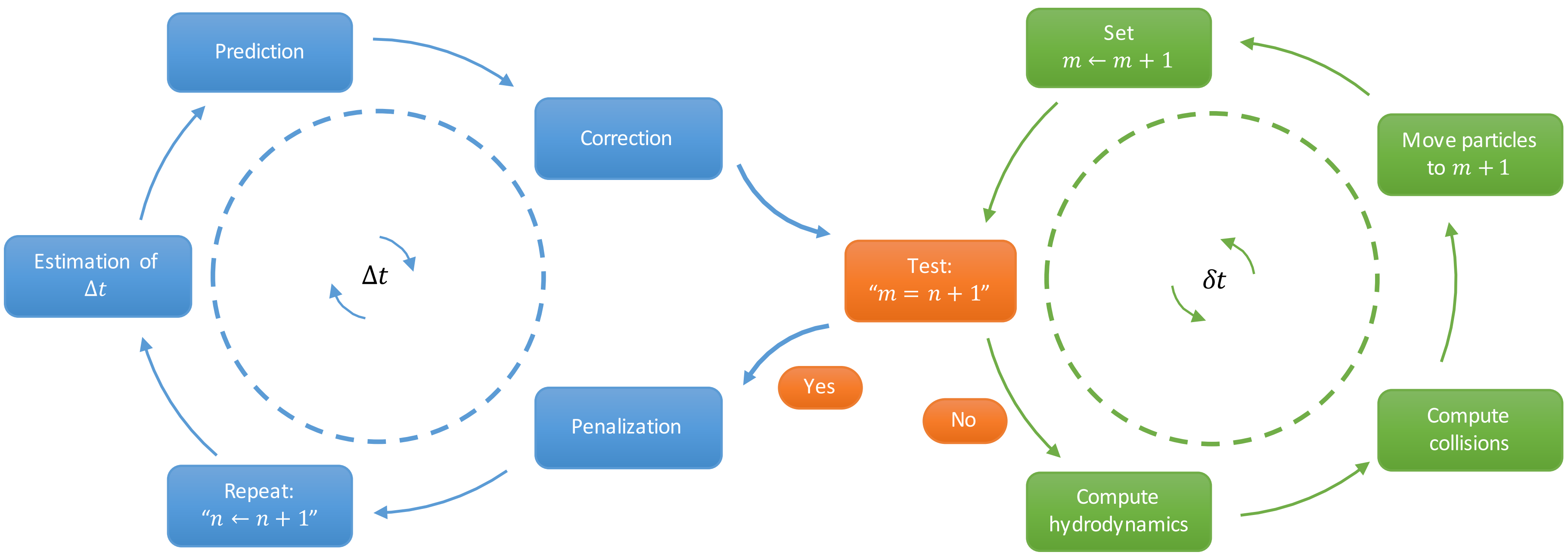


Summary: Numerical Resolution

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} &= -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\Delta\mathbf{u} + \frac{1}{\rho}\mathbf{F}^{\mathbf{ext}} + \chi\lambda\,(\mathbf{u}_s - \mathbf{u}) \\ \nabla\cdot\mathbf{u} &= 0 \end{cases}$$

$$m_i \frac{d\mathbf{U}_i}{dt} = \mathbf{F}_i^{hyd} + \mathbf{F}_i^{coll} + \mathbf{F}_i^{ext} + \mathbf{F}_i^{lub}$$

$$\overline{\overline{I}}_i \frac{d\Omega_i}{dt} = \mathbf{T}_i^{hyd} + \mathbf{T}_i^{coll} + \mathbf{T}_i^{lub}$$

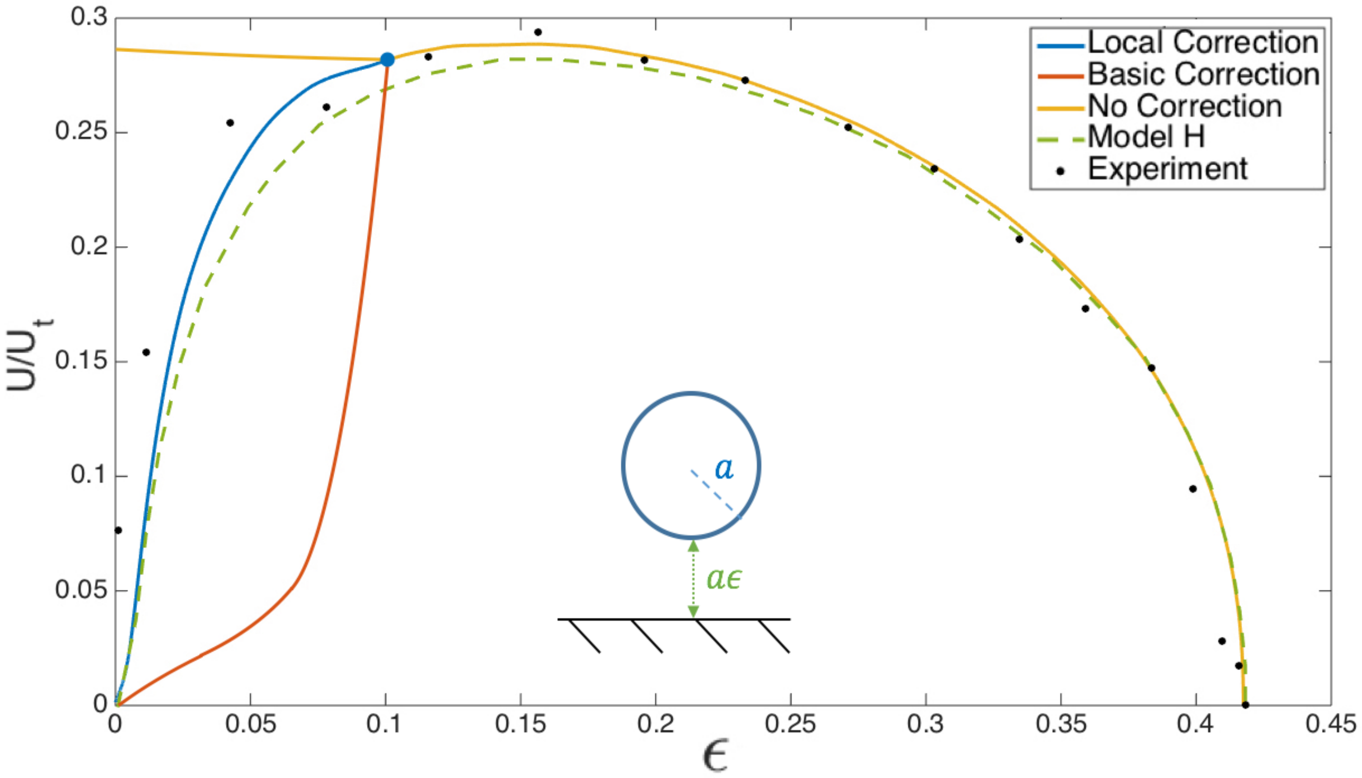


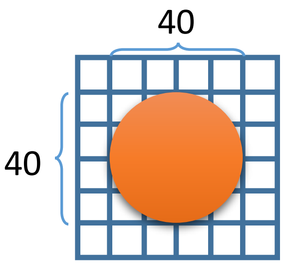
The Local Lubrication Correction Model

Some Numerical Results

Conclusion

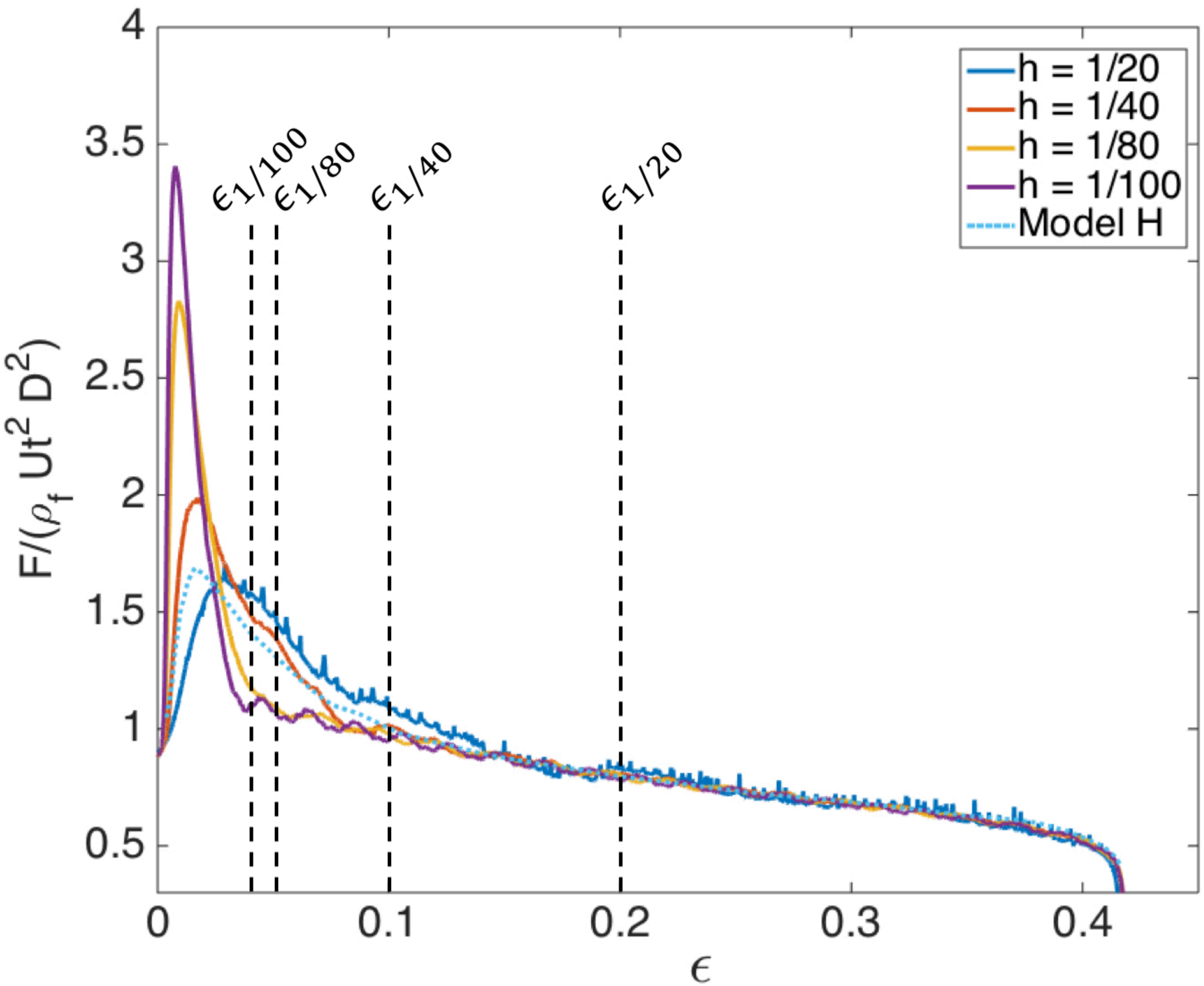
Falling Particle: Velocity Profile

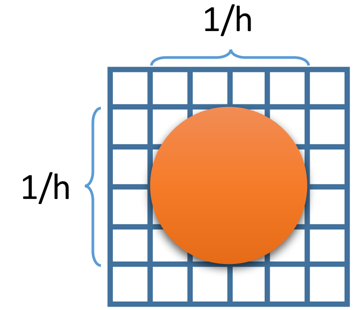


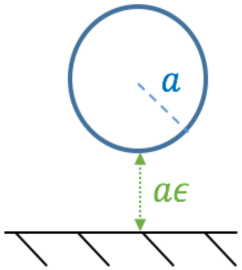


Reference: Harada *et. al.* (2001)

Falling Particle: Grid Sensitivity

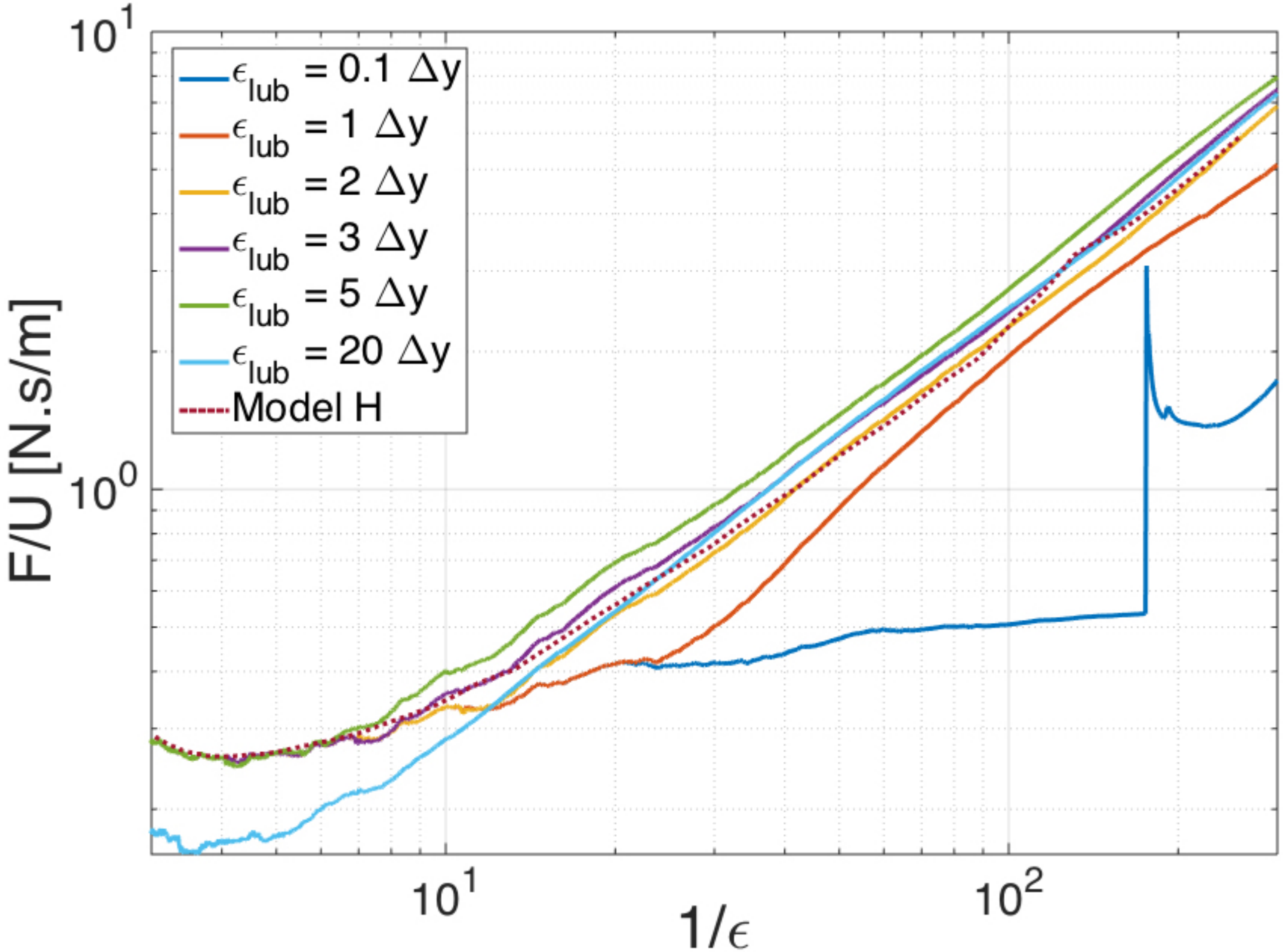


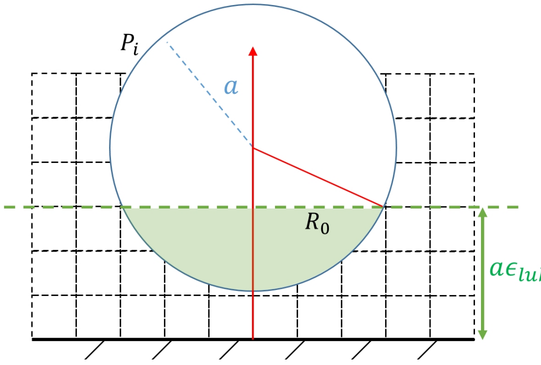




Reference: Harada *et. al.* (2001)

Falling Particle: Sensitivity to the Lubrication Parameter





$$\frac{\mathbf{F}_i^{lub}}{\pi \mu a U_i} = -\frac{6}{\epsilon}\mathbf{e_z} + O(1)$$

Reference: Harada *et. al.* (2001)

Dam Breaking: Accuracy…